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Evolution, efficiency and noise traders in a one-sided auction market ☆

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Abstract

Natural selection is used to examine a one-sided buyer auction market. With each trader's behavior preprogrammed with its own inherent and fixed probabilities of overpredicting, predicting correctly and underpredicting the fundamental value of the asset, informational efficiency occurs. If each buyer's initial wealth is sufficiently small relative to the market supply and if the variation in the asset's random shock is sufficiently small, then as time gets sufficiently large, the proportion of time, that the asset price is arbitrarily close to the fundamental value, converges to one with probability one. This is established under a weak restriction regarding traders' behavior. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Traditionally, in the literature, the derivation of an informationally efficient market has tended to rely on the presence of traders' rational expectations, strategic

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usage of market information or adaptive learning behavior where noise traders gradually become informed traders. However, due to individuals' limited ability to process and manage complex information, the assumption of rationality is challenged. This further calls into question the achievement of market efficiency. On the other hand, there is Friedman's (1953) well-known conjecture that, because noise traders will sooner or later lose money to the informed traders, the informed traders will come to dominate the market and drive the asset price toward the fundamental value. An intuitively appealing aspect of Friedman's conjecture is the idea of natural selection among traders.

The idea of abandoning rationality on the part of traders is consistent with a growing literature in behavioral economics and finance. The behavioral approach focuses on the behavior patterns drawn from psychological theory (e.g., Kahneman et al., 1982). Often judgmental decisions are based on cognitive rules of thumb used to simplify the decision-making process. Using such rules to assess uncertain events and make predictions, often leads to systematic errors or biases. In the context of a one-sided auction, such systematic errors take the form of consistent patterns of predicting biases, which are captured by the probabilities of overpredicting and underpredicting the asset value; these predicting probabilities become the key to modeling traders' behavior. In this paper it is assumed that traders are rather unsophisticated and each trader consistently overpredicts or underpredicts with some fixed probabilities. In an evolutionary sense, each trader is genetically preprogrammed with its own inherent and fixed probabilities of overpredicting, underpredicting and predicting correctly the asset value. Since a trader has a positive probability of overpredicting or underpredicting the asset value, the trader has a positive probability of acting upon noise as if it were information. Therefore, in this sense, traders are called noise traders in this paper.

Within the context of a double-sided auction, Luo (1998, 2001) shows that with no requirement of traders' rationality such as rational expectations and adaptive learning, natural selection among traders through redistribution of wealth is sufficient to cause the convergence to an informationally efficient market. It is noteworthy, however, that in the context of a double-sided auction, the allowance for short sales implies that the supply of the asset is virtually elastic. While the majority of financial markets adopts a double-sided auction market, there is a significant number of markets which are essentially one-sided auction markets with a perfectly inelastic supply. A key distinction between a one-sided and a double-sided auction market is that traders in a double-sided auction market are allowed to short sell. In a one-sided auction market, short sales are not possible because of the lack of a secondary market. The absence of secondary markets often occurs in the sale of short-term commercial paper, municipal notes, non-negotiable certificates of deposit and private placements and sometimes occurs in markets for bonds (e.g., Japanese corporate bonds).[1] Other examples of one-sided auctions include the leasing of mineral rights, the leasing of oil drilling permits and the leasing of timber

[1] Sometimes, secondary markets do exist for commercial paper and private placements. But usually the transaction costs of setting up these markets have proven to be prohibitive.

rights.[2] The recent emergence of internet online purchasing provides some other interesting examples.

There are three purposes of this paper. The first purpose is to show that the market mechanism of a one-sided auction, itself, can promote an efficient outcome through natural selection. To this end, this paper adopts the idea of natural selection and formulates an evolutionary model of a one-sided buyer's auction market. It shows that even without a rationality assumption like rational expectations or adaptive learning and even when each trader merely acts upon its own inherent and fixed probabilities of overpredicting, predicting correctly and underpredicting the asset value, an informationally efficient market can occur. This stands in contrast with the conventional literature which states that an efficient outcome is promoted through rational expectations or adaptive behavior on the part of market participants (e.g., Grossman, 1976, 1978; Radner, 1979; Hellwig, 1980).

A second purpose of this paper is to identify, in the context of this evolutionary approach, an alternative less restrictive condition for achievement of an efficient outcome in the one-sided auction market other than the one used in the double-sided auction market. For convergence to occur in the context of a double auction, Luo (1998, 2001) requires that in each time period there is a positive probability that the entering trader has an arbitrarily high probability of predicting the fundamental value correctly. Furthermore, within the context of a double auction the allowance for short sales implies that the supply of contracts is elastic. In contrast, in this paper, the supply of the asset is perfectly inelastic and the market is a one-sided buyer auction. For convergence to occur, a less restrictive condition is that there is a positive probability in each time period that the entering trader has an arbitrarily low probability of overpredicting the fundamental value and has a probability, of predicting arbitrarily close to the fundamental value, being bounded away from zero by a positive number. This result is very intuitive. If traders who overpredict with a low probability (implying a small number of upward biases) are characterized as being relatively risk averse (in a behavioral finance sense (e.g., Sitkin and Pablo, 1992), for convergence to occur, it is sufficient that there are enough traders with a sufficiently degree of risk aversion. The importance of the presence of these risk averse traders in the natural selection process leading to market efficiency, is consistent with Barrow (1992) and Olsen (1998) who see decision attributes that exhibit aversion to negative impacts (here caused by overpredicting) as having evolutionary value in selecting the long run outcome.

Nevertheless, to obtain convergence to efficiency, competition among traders is needed. That is, in a one-sided auction market where the market supply is perfectly inelastic, competition means that each trader's initial wealth must be sufficiently small relative to the fixed market supply. However, it is remarkable that even when there is a perfectly inelastic supply of the asset and even when the number of traders increases over time, with each entering trader coming in with a finite amount of

[2] Other examples of auctions, where there is a fixed supply of an asset and no short sales, can be found in both the theoretical and experimental economics literature (e.g., Wilson, 1979; Forsythe et al., 1982).

wealth, the price can be eventually assured to remain in a small interval containing the fundamental value.

A third purpose of this paper is to quantify the extent to which a variation in movement of the underlying asset's liquidation value around the fundamental value influences the extent of convergence of the asset price to the fundamental value. This paper, like Luo (2001), accounts for a random shock around the fundamental value. This paper shows that, the bigger is the variation in the random shock about the fundamental value, the bigger is the variation of the asset price about the fundamental value.

The evolutionary approach in this paper stands in contrast with recent literature which has analytically studied the impact of noise traders on the market price within the context of rational models. For example, De Long et al. (1990) study an overlapping generation model, where an informed trader and a noise trader both maximize their expected utilities. They find that the noise trader can cause the market price to deviate systematically away from its fundamental value.[3] For the convergence of the asset price to the fundamental value, rationality models (where utility or profit maximization occurs) sometimes rely on learning and imitation to make uninformed traders become more informed. For example, Grossman (1978) allows investors to acquire information about past distributions of prices which in turn produces more informed investors. This has also been explored in the experimental laboratory market. Plott and Sunder (1982) show that constant replication allows a learning by traders which resembles a rational expectations equilibrium.

Related literature in applying an evolutionary approach to examining market behavior are Blume and Easley (1992), Biais and Shadur (1993) and Luo (1995). Blume and Easley (1992) study a dynamic model of an asset market with a finite number of traders with different investment rules. They find that the market can select for an irrational rule and the market may not be efficient. Biais and Shadur (1993) apply Darwinian dynamics to the selection of the number of informed traders and noise traders based on their payoffs. They find that noise traders can persist in the long run. The results in the above two papers have illustrated that the natural selection in the market alone is not sufficient to generate an efficient market outcome. Luo (1995) further shows that natural selection, in conjunction with competition in the market, is sufficient to lead the market to select the most efficient firms (those producing at or near the minimum efficient scale) and as a result the market price converges to the perfectly competitive price.

This paper shows analytically that the market can reach an efficient market outcome with the presence of irrational noise traders. The coexistence of market efficiency and participants' irrationality has also been illustrated in some recent experimental literature as well. For example, Gode and Sunder (1993) design a series of experiments to examine a double auction market where traders submit bids and offers randomly. They find that allocation efficiency in the double auction can be

[3] Their results are partly due to the difference in the utility functions of both types of traders, the absence of wealth accumulation and the absence of wealth flows between the two types of traders.

generated from individual participants' irrationality. Bosch and Sunder (1994) design a similar series of experiments to examine multimarket double auctions. They reach the same conclusion.

This paper is organized into four sections. Section 2 outlines the model. The results can be found in the Section 3. Section 4 summarizes and concludes the paper.

2. The model

Consider a conventional dynamic asset market where a market maker supplies one unit of a one-period risky asset each time period. Time is indexed by t, where $t = 1, 2, \ldots$. The asset market opens each time period. Traders can only buy shares of the asset from the market maker at the market clearing asset price. Denote the market clearing price at time s as p_s, where $s = 1, 2, \ldots$. No short selling is possible in this framework. Hence, all traders are buyers. In the following sections, the terms traders and buyers are used interchangeably. The risky asset is liquidated after the asset market closes at the end of each time period.[4] Denote v_t as the liquidation value of the risky asset at the end of time period t. It is assumed that $v_t = z_t \exp(w_t)$, where z_t represents the fundamental liquidation value of the asset at time t and determined in the beginning of time t and where w_t is a random shock to the economy and realized at the end of each time period t. It is assumed that the sequence $\{z_t\}_{t \geqslant 1}$ is a random sequence taking values from an interval $[\underline{z}, \bar{z}]$, where $0 \leqslant \underline{z} \leqslant \bar{z} < \infty$. The sequence $\{w_t\}_{t \geqslant 1}$ is assumed to be an i.i.d random sequence taking values from an interval $[-\underline{w}, \bar{w}]$, where $0 < \underline{w} < \infty$, $0 < \bar{w} < \infty$ and $E(w_t) = 0$ for all t. *It is assumed that ω_t, where $t = 1, 2, \ldots$, is independent of z_s, where $s \geqslant 1$.*

2.1. Traders' prediction errors

Traders enter the market sequentially over time. At the beginning of time period t, where $t = 1, 2, \ldots$, the fundamental value z_t is determined, but it is unknown to all market participants. At the beginning of time period t after the fundamental value z_t is determined, only one trader (called trader t) is allowed to enter the market and participate along with previously entered traders in the asset market. Trader t has a prediction (or belief) about the fundamental liquidation value each time period. Denote trader t's prediction (or belief) about the fundamental liquidation value at time s as b_s^t, where $s \geqslant t$. Define trader t's prediction error with respect to the fundamental liquidation value at time s ($s \geqslant t$) as $u_s^t = b_s^t - z_s$.[5] *It is assumed that u_s^t is independent of $z_{s'}$, where $s' \geqslant 1$, and for all $s \geqslant t$, $u_s^t \in [-u, u]$,*[6] *and furthermore, for trader $t = 1, 2, \ldots$, u_s^t is independently and identically distributed across times $s = t + 1, t +$*

[4] This one-sided buyer market with one unit of an asset being supplied each time period is also used in Blume and Easley (1992).

[5] This specification of beliefs or predictions is consistent with Grossman (1976, 1978), Figlewski (1978) and Hellwig (1980).

[6] For negative predictions not to occur, the lower bound for predictions is constrained to zero. That is $b_s^t = \max[z_s + u_s^t, 0]$.

2, ... Hence, for any given positive ε, denote

$$\theta_1^t = \Pr(u_s^t > -\tfrac{1}{3}\varepsilon), \quad \theta_2^t = \Pr(-\tfrac{2}{3}\varepsilon \leqslant u_s^t \leqslant -\tfrac{1}{3}\varepsilon) \quad \text{and} \quad \theta_3^t = \Pr(u_s^t < -\tfrac{2}{3}\varepsilon),$$

where superscript t indicates buyer t. In other words, the variables θ_1^t, θ_2^t and θ_3^t are trader t's probabilities of overpredicting, predicting correctly and underpredicting the fundamental value. It is assumed that trader t's prediction error distribution characteristics (θ_1^t, θ_2^t) are randomly taken from the set $\Theta = \{(\theta_1, \theta_2) \in (0,1) \times (0,1)\colon \theta_1 + \theta_2 < 1\}$ according to a given distribution function $F(\cdot)$ in the beginning of this trader's entry period and $\theta_3^t = 1 - \theta_1^t - \theta_2^t$. Once trader t's vector $(\theta_1^t, \theta_2^t, \theta_3^t)$ is determined in the beginning of trader t's entry period, it is fixed thereafter in any subsequent time period. Fig. 1 shows the timing of the above events.

The closer to zero a prediction error is, the more accurate the information is reflected in the prediction. This way of modeling the trader's beliefs reflects the fact that different traders may receive different signals about the fundamental value of the asset; or that, even though they receive the same signal, they may have different abilities to process the same information. From a behavioral finance perspective, the prediction error can be interpreted as a judgmental bias in the decision-making process. Based on the psychology literature, this judgmental bias tends to be displayed in a systematic pattern through time (see Tversky and Khaneman (1974) and Proposition 2 regarding inertia of Sitkin and Pablo (1992)). For this reason, a trader's probabilities of overpredicting, predicting correctly and underpredicting is modeled as being fixed through time. In contrast to some of the literature (e.g., Grossman, 1976, 1978; Figlewski, 1978; Hellwig, 1980), no precise assumptions are made about the shape of the distribution of the prediction error or the mean or variance of the prediction error.

Trader t's prediction or belief at time s, where $s \geqslant t$, is generated from trader t's prediction error probability distribution characterized by the vector $(\theta_1^t, \theta_2^t, \theta_3^t)$. Trader t's prediction or belief b_s^t at time s is this trader's reservation price or bid, indicating that trader t is willing to buy the number of shares of the asset that his or her wealth permits at a price no higher than b_s^t at time s. For this reason, the words prediction and bid are used interchangeably.

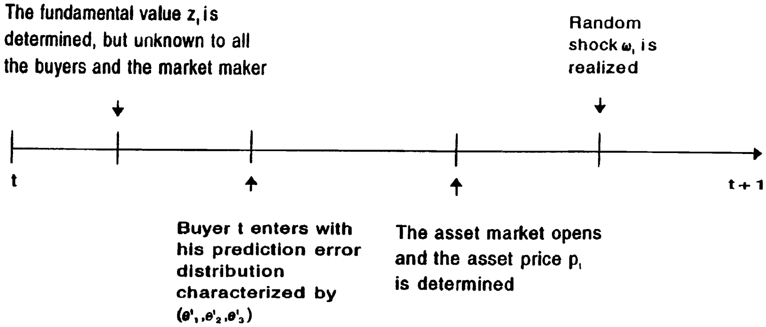

Fig. 1.

Here, there is no strategic interaction or learning among traders taking place. Traders merely act upon their predetermined probabilities of overpredicting, predicting correctly and underpredicting the fundamental value. Using biological language, one may think of the vector $(\theta_1^t, \theta_2^t, \theta_3^t)$ as trader t's phenotype. This way of modeling traders' behavior isolates the impact of the market selection on traders' wealth dynamics and highlights the role that market selection plays in promoting long run market efficiency. Since each trader has a positive probability to underpredict or overpredict the fundamental liquidation value, each trader has a strictly positive probability of acting upon noise as if it were information.

For the simplicity, the following assumptions are used in this paper.

It is assumed that $\{(\theta_1^t, \theta_2^t)\}_{t\geqslant 1}$ *is independently and identically distributed according to the distribution function* $F(\cdot)$ *across traders* $t = 1, 2, \ldots$.

It is also assumed that trader i's prediction error at time s, u_s^i, *where* $s \geqslant i \geqslant 1$, *is independent of trader j's prediction error at time* s', $u_{s'}^j$, *where* $s' \geqslant j \geqslant 1$ *and* $j \neq i$.

Furthermore, it is assumed that the random shock at time s, ω_s, *where* $s \geqslant 1$, *is independent of trader t's prediction error at time* s', $u_{s'}^t$, *where* $s' \geqslant t \geqslant 1$.

Finally, define all the possible states of the world at time period t, where $t \geqslant 1$, as $\Omega_t = [-u, u]^t \times [\underline{z}, \bar{z}] \times [-\underline{\omega}, \bar{\omega}]$ with a typical element $\xi_t = (u_t^1, u_t^2, \ldots, u_t^t, z_t, \omega_t)$. ξ_t is the state of the world at time t. Denote $\Omega = \prod_{t=1}^{\infty} \Omega_t$. Let $\Im$ denote the product σ-field on Ω and let $\Pr(\cdot)$ denote the product probability. The probability space is $(\Omega, \Im, \Pr(\cdot))$. Let $\Omega^t = \prod_{i=1}^{t} \Omega_i$ for $t \geqslant 1$. Then the typical element in the set Ω^t is $(\xi_1, \xi_2, \ldots, \xi_t)$.

2.2. Traders' wealth dynamics in the market process

Each trader is assumed to have initial endowment of wealth V_0 in his or her entry time period. Each trader's wealth at the end of a time period is defined as accumulated profits up to the end of that time period. Denote trader t's wealth at the end of time period s (where $s \geqslant t \geqslant 1$) as V_s^t and $V_{t-1}^t = V_0$.[7] If trader t's prediction or bid at time s, where $s \geqslant t$, is b_s^t, then trader t's demand for shares of the asset in time period s, where $s \geqslant t$, q_s^t, is[8]

$$q_s^t(p_s) = \begin{cases} \dfrac{V_{s-1}^t}{p_s}, & p_s < b_s^t, \\ \left[0, \dfrac{V_{s-1}^t}{p_s}\right], & p_s = b_s^t, \\ 0, & p_s > b_s^t. \end{cases}$$

[7] Alternatively, one can assume different initial endowments of wealth for all traders. However, the same results with respect to convergence of the asset price will hold, provided that the initial endowment of wealth for all entering traders is bounded from above.

[8] For simplicity it is assumed that each trader potentially invests all of his or her wealth in each period. The results of this paper would still hold if each trader spends only a fraction of his or her total wealth on trading activity. A smaller fraction could reflect more risk aversion on the part of the trader.

This demand function implies that if the asset price is below trader t's prediction, this trader is willing to purchase the number of shares that this trader's wealth permits; if the asset price is above trader t's prediction, this trader is not willing to purchase any share of the asset; if the asset price is equal to trader t's prediction, this trader is indifferent in purchasing any number of share lying between 0 and V_{s-1}^t/p_s.

A trader is said to be active at one time period if the trader has actually purchased any shares at that time period. Therefore, if $q_s^t > 0$ then trader t is active in the time period s. If $q_s^t = 0$ then trader t is not active at time period s. A trader's profit at the end of time period s is $(v_s - p_s)q_s^t$. Hence, at the end of time period s, the wealth of trader t is[9] $V_s^t = V_{s-1}^t + (v_s - p_s)q_s^t$.[10]

2.3. The asset market equilibrium

The asset market mechanism is organized as a one-sided auction market structure. The asset price is merely a Walrasian market clearing price. The asset market is assumed to have no traders at time 0. After time 0 traders enter sequentially. Since the traders' aggregate demand for the asset at time s is $\sum_{t=1}^{s}(q_s^t(p_s))$ and since the market supply of the asset in each time period is 1 unit, the market clearing price at time s is the solution to the equation $\sum_{t=1}^{s}(q_s^t(p_s)) = 1$.

Clearly, the solution to the above equation is unique due to the shape of the traders' demand functions and the vertical supply function of the asset from the market maker.

All transactions are executed at the market clearing price. Traders with reservation bids or predictions above p_s will exhaust their wealth at time s. If the number of shares of the asset demanded at the market clearing price exceeds the total supply of the asset; and if there are traders whose reservation bids or predictions coincide with the market clearing price, the market maker allocates the supply of the asset among those traders proportionately to their wealth at the market clearing price.[11] Therefore, all traders with reservation bids equal to p_s at time s may not exhaust all of their wealth.

As can be seen, the asset price p_s is a function of the realizations of $z_1, z_2, \ldots, z_s$; and $\omega_1, \omega_2, \ldots, \omega_{s-1}$; and $u_1^t, u_2^t, \ldots, u_s^t$, for all $t \leqslant s$; and V_0. Hence, over time, the asset price follows a very complicated stochastic process.

In this one-sided auction market, the market serves as a selection process that evaluates all the traders with different prediction error distribution characteristics, rewarding traders who place good bids with gains (more wealth) and punishing

[9] It is apparent that, given the demand function, any trader's wealth can never be negative. This is due to the fact that each trader begins with a positive level of initial wealth and short sales are not allowed.

[10] A related paper which examines the relationship between accuracy of agents' predictions, wealth accumulation and the convergence of prices to the true value, is Sandroni (2000).

[11] That is, for trader t where

$$b_s^t = p_s, \quad q_s^t = \frac{V_{s-1}^t}{\sum_{t' \in M_s} V_{s-1}^{t'}}\left[1 - \sum_{t' \in A_s} q_s^{t'}\right]$$

where $A_s = \{t' \mid b_s^{t'} > p_s\}$ and $M_s = \{t' \mid b_s^{t'} = p_s\}$. It would not alter the results if any alternative method of allocating shares of the asset is used in this situation.

traders who place bad bids with losses (less wealth). The more wealth a trader has, the more influence this trader has over the asset price. All traders impact the asset price through their predictions and the wealth they possess.

This paper now uses the above evolutionary model of an asset market, to examine the relationship between the asset price and the fundamental value. In this exploration, one assumption is made about the distribution function $F(\cdot)$ from which $(\theta_1^t, \theta_2^t, \theta_3^t)$ for all t are drawn. This assumption says that in each time period there is a positive probability that an entering trader has an arbitrarily low probability of overpredicting the fundamental value and has a probability, of predicting arbitrarily close to the fundamental value, being bounded away from zero by a positive number. Since this assumption characterizes the distribution of traders' thetas ($(\theta_1^t, \theta_2^t, \theta_3^t)$ for all t), this assumption is referred to as the θ-Assumption in the remainder of the paper. More precisely,

θ-Assumption. For any given small positive η and η',

$$\int_{\substack{0<\theta_1<\eta,\\ \theta_2>\eta'}} \mathrm{d}F(\theta_1, \theta_2) > 0.$$

This is not a very demanding assumption. It merely says that with a positive probability (this probability could be extremely small), in each time period, the entering trader overpredicts less often (as a result, making losses less often) than any of previously entered traders. Note that the θ-Assumption does not mean that all entering traders must overpredict less often than any of previously entered traders. The key phrase in this assumption is "there is a positive probability" (this probability could be very small) that the entering trader in each time period overpredicts less often than any of previously entered traders.[12]

Since in this one-sided auction, losses are possibly realized when overbidding occurs, traders who overpredict with a sufficiently small probability can be viewed as avoiding risk of making losses due to placing a high weight on negative outcomes; hence, in the behavioral sense, such traders can be referred to as relatively risk averse (see Schneider and Lopes, 1986; Sitkin and Pablo, 1992). For this reason, one may interpret this assumption as an allowance for the entry of a sufficient number of buyers with a sufficient degree of risk aversion, who have a positive probability (however small) of bidding arbitrarily close to the fundamental value.

3. Convergence of the asset price to the fundamental value

This section shows analytically that with probability 1, if each trader's initial wealth is sufficiently small relative to the market asset supply and if the variation in the random shock is sufficiently small, then as time gets large, the proportion of time,

[12] In fact, there can be a positive probability of traders with other types of predictive behavior. For example, such trader types could enter with prediction behavior that imitates past successful traders (e.g., Lettau, 1997). Nevertheless, as long as the θ-Assumption holds, the presence of these types of traders would not disrupt convergence.

that the asset price lying in an arbitrarily small interval, containing the fundamental value, converges to 1. This is shown in Theorem 3.

Before Theorem 3 is proven, two theorems are needed. Theorem 1 shows that the proportion of time, that the asset price is below the fundamental value by at least a small positive number ε, converges to zero with probability 1. Theorem 2 shows that if each trader's initial wealth is sufficiently small relative to the market asset supply and if the variation in the random shock to the economy is sufficiently small, then as time gets sufficiently large, the proportion of time, that the asset price is above the fundamental value by at least a small positive number ε, converges to zero with probability 1. Theorem 3 follows directly from Theorem 1 and Theorem 2. The result in Theorem 3 further suggests that in the limit, each individual trader's demand must be infinitesimally small relative to the market supply. This is formally established in Corollary 1.

For the establishment of the result in Theorem 1, the θ-Assumption plays a critical role. It allows enough "better buyers" to enter the market over time and eventually those "better buyers" prevent the asset price from staying below the fundamental value very often. The "better buyers" here refer to the buyers who enter the market with a sufficiently low probability of overpredicting the fundamental value and with some positive probability (which may be very small) of predicting arbitrarily close to the fundamental value.

Theorem 1.

$$\Pr\left(\textit{for any given } \varepsilon > 0, \textit{ there exists a time period } T_1,. \right.$$
$$\left. \textit{such that for } T \geqslant T_1, \frac{\#\{t \leqslant T\colon p_t < z_t - \varepsilon\}}{T} < \varepsilon \right) = 1.$$

Proof. See Appendix A for the proof. □

After obtaining the result in Theorem 1, Theorem 2 shows that with probability 1 the following occurs: if each trader's initial wealth is sufficiently small relative to the market asset supply and if the variation in the random shock to the economy is sufficiently small, then as time gets sufficiently large, the proportion of time, that the asset price is above the fundamental value, converges to zero.

The intuition of the result in Theorem 2 is best explained as follows. Start by supposing that there is no random shock to the risky asset each time period. Then the liquidation value is the same as the fundamental value. In this case, buyers make a gain if the asset price is below the fundamental value and buyers make a loss if the asset price is above the fundamental price. Therefore, the average (across time) aggregate wealth of all buyers is bounded from above by term (1), the average maximum gains across the number of time periods when the asset price is below the fundamental value, minus term (2), the average minimum loss across the number of time periods when the asset price is above the fundamental value, plus term (3), the initial wealth V_0. Now suppose that there is a random shock each time period. Then the liquidation value differs from the fundamental value. If the random shock is positive (negative) at time t, $(t \geqslant 1)$ then the liquidation value is above the fundamental value at time t and furthermore all buyers make an additional gain (loss) from the difference between the liquidation value and the fundamental value $v_t - z_t$ at time t as a result of the presence of the random shock. Therefore, the average (across time) aggregate wealth of all buyers is bounded from above by the average aggregate gains of all buyers coming from the positive random shock in addition to the above three terms describing the situation without the random shock.

Furthermore, the aggregate wealth of all buyers is always positive. Hence, we can rearrange the above to show that the proportion of time that the asset price is above the fundamental value is bounded from above by the summation of three terms: the first term, which is a positive constant multiplied by the proportion of time that the asset price is below the fundamental value; the second term, which is a product of the variation in the random shock (given that the shock is positive) and the proportion of time that the random shock is positive; and a third term, which is V_0. Theorem 1 says that, with probability one, the proportion of time, that the asset price is below the fundamental value, goes to zero as time gets large. As well, if the variation in the random shock is sufficiently small and the initial wealth is sufficiently small, then the proportion of time that the asset price is above the fundamental value can be made arbitrarily small, with probability one.[13] Hence, the result of Theorem 2 holds.

Theorem 2. With probability 1, *the following occurs*: *for any given* $\varepsilon > 0$, *there exists a positive* $\bar{k}$ *and a positive* $\bar{V}_0$ *such that*, *if* $\mathrm{E}((\exp(\omega_t) - 1)|\omega_t > 0) \leqslant \bar{k}$ *and if* $V_0 \leqslant \bar{V}_0$, *there exists a time period* T_2, *such that for all* $T > T_2$,

$$\frac{\#\{t \leqslant T\colon\, p_t > z_t + \varepsilon\}}{T} < \varepsilon.$$

Proof. See Appendix A for the proof. □

Theorem 3 brings together the results of the above two theorems and shows that for any ε-interval containing z_t, $[z_t - \varepsilon, z_t + \varepsilon]$, for a sufficiently small endowment of traders' initial wealth and for a sufficiently small amount of variation in the random shock, with probability one, the proportion of times that the asset price falls within this ε-interval containing the fundamental value z_t, $[z_t - \varepsilon, z_t + \varepsilon]$, converges to one as time gets large.

Theorem 3. With probability 1, *the following occurs*: *for any given* $\varepsilon > 0$, *there exists a positive* $\bar{k}$ *and a positive* $\bar{V}_0$ *such that*, *if* $\mathrm{E}((\exp(\omega_t) - 1)|\omega_t > 0) \leqslant \bar{k}$ *and if* $V_0 \leqslant \bar{V}_0$, *there exists a time period* T^*, *such that for all* $T > T^*$,

$$\frac{\#\{t \leqslant T\colon\, p_t \in [z_t - \varepsilon, z_t + \varepsilon]\}}{T} > 1 - 2\varepsilon.$$

[13] The variation in random shock here refers to $\mathrm{E}((\exp(\omega_t) - 1)|\omega_t > 0)$.

Proof. See Appendix A for the proof. □

Remark 1. The proof of Theorem 1 makes use of a set of buyers $\{t_1, t_2, \dots, t_k, \dots\}$ with their prediction errors distribution satisfying the following conditions $\varepsilon_0 \theta_2^{t_k} \ln(\bar{z}/(\bar{z} - \frac{2}{3}\varepsilon)) + \theta_1^{t_k} \ln(\underline{z}/(\bar{z} + u)) > 0$ and $\theta_2^{t_k} \geqslant c$ for some positive ε_0 and $c < 1$. These conditions are satisfied if $\theta_1^{t_k}$ is allowed to be arbitrarily close to zero provided that $\theta_2^{t_k} \geqslant c$ (the θ-Assumption). (Of course, the constant number c does not have to be very close to 1, instead, it can be a extremely small positive number.) This set of buyers play a determining role in driving the convergence result. The model deals with a one-sided, rather than a two-sided auction. With a relatively small variation in the random shock, buyers do not make any loss by underpredicting the fundamental value. This is due to buyers being inactive if the asset price is above the fundamental value. Buyers could make losses by overpredicting the fundamental value if the asset price is above the fundamental value. From a behavioral finance perspective, one can interpret traders with low probability of overpredicting the fundamental value (implying a small number of upward biases) as relatively risk averse. For this reason, the θ-Assumption merely allows for the entry of a sufficient number of buyers with a sufficient degree of risk aversion and with a positive probability (however small) of bidding arbitrarily close to the fundamental value. The presence of this group of buyers provides the driving force for the long run convergence.

Remark 2. For the results to be achieved it is important that the competition among the buyers gets stronger as time increases. The model helps to achieve this by letting buyers with different characteristics of their prediction error distributions enter the market sequentially over time. As a result, the number of buyers increases to infinity as time goes to infinity.

Another ingredient of perfectly competitive markets is that each individual buyer is sufficiently small relative to the market or that each individual buyer's demand is initially sufficiently small relative to the market supply. In this model, where the initial wealth V_0 characterizes the size of the entering buyer relative to the market, a smaller V_0 represents less market power of entering traders and greater competition in the market. Technically, if the initial wealth V_0 is too large then in each time period the entering trader will cause the asset price to deviate away from its fundamental value with some positive probability.

Remark 3. Theorem 3 states that for a given ε-distance, for a sufficiently small V_0 and a sufficiently small variation in the random shock, with probability one, the proportion of time, that the deviation between the asset price and the fundamental value lies within this ε-distance, converges to one as time gets large. An empirical implication is, that for a given V_0, with a smaller variation in the random shock, there exists a corresponding smaller ε_1-distance, for which, with probability one, the proportion of time, that the deviation between the asset price and the fundamental value lies within this smaller ε_1-distance, converges to one as time gets large. This is consistent with the price of a more risky asset fluctuating in a wider range than the price of a less risky asset and the safe asset earning the certain return.

Nevertheless, it is remarkable that even though the number of traders increases over time, with each trader entering the market with a finite amount of wealth V_0 and with a perfectly inelastic supply of the asset, the price can eventually be assured never to explode outside a certain interval. Furthermore, for a given level of variation in random shock, the smaller is the initial V_0 of an entering trader, the smaller is the interval in which the asset price lies.

The result of Theorem 3 indicates that in the limit no individual buyer can influence the asset price. Otherwise, the long-run equilibrium would be destroyed. This further implies that the demand from each individual buyer must be infinitesimally small relative to the asset market supply in the limit. In other words, each individual buyer's wealth must be infinitesimally small relative to the aggregate wealth of all the buyers in the limit. The following corollary formally presents the results.

Corollary 1. With probability 1, *the following occurs: for any given* $\varepsilon' > 0$, *there exists a positive* $\bar{k}$ *and a positive* $\bar{V}_0$ *such that, if* $\mathrm{E}((\exp(\omega_t) - 1)|\omega_t > 0) \leqslant \bar{k}$ *and if* $V_0 \leqslant \bar{V}_0$, *there exists a time period* T' *such that for all* $T > T'$, *the ratio between buyer* t*'s* $(t = 1, 2, \ldots)$ *wealth at the end of time period* T (V_T^t) *and the aggregate wealth of all the buyers at the end of time period* T $(\sum_{t=1}^{T} V_T^t)$ *is less than* ε' (i.e., $V_T^t / \sum_{t=1}^{T} V_T^t \leqslant \varepsilon'$).

Proof. See Appendix A for the proof. □

4. Conclusions

This paper uses an evolutionary idea of natural selection to examine the convergence of the asset price to its fundamental value in a one-sided auction market where traders are modeled as unsophisticated. Traders cannot learn or strategically use any information from the market or other participants and they merely act upon their preprogrammed behavior rules, which reflect their inherent noisiness. Specifically, in this dynamic asset market, the market supplies one unit of a one period risky asset each time period. Traders enter the market sequentially over time. The liquidation value of the asset is the product of the fundamental value of the asset and the exponential of a random shock. The fundamental value of the asset is determined in the beginning of each time period, but unknown to any market participants. The random shock is realized at the end of each time period. Each trader's behavior is exogenously preprogrammed instead of endogenously derived from some optimization problem. In other words, each trader's behavior is characterized by his or her predetermined probabilities of overpredicting, predicting correctly and underpredicting the fundamental liquidation value. Each trader's prediction in each time period is generated from a probability distribution described by his or her predetermined probabilities of overprediction, exact prediction and underprediction with respect to the fundamental liquidation value.

The asset market is a one-sided auction market. The market serves as a selection process which evaluates all the traders with different prediction error distributions, rewarding the traders who place good bids and taking money away from the traders who place bad bids. Over time this process gradually places more weight on the accurate information into the asset price and places less weight on noise into the asset price. Eventually the asset price only reflects the accurate information and eliminates noise. Informational efficiency occurs in the long run.

More precisely, as long as at any point in time there is a positive probability that the entering trader has an arbitrarily low probability of overpredicting the fundamental value and has a probability, of predicting arbitrarily close to the fundamental value, being bounded away from zero by a positive number, the following is true. With probability 1, if each trader's initial wealth is sufficiently small relative to the market supply and if the variation of the random shock is sufficiently small, then as time gets sufficiently large, the proportion of time, that the asset price is arbitrarily close to the fundamental value, converges to 1. In the limit, each individual's demand for the risky asset is infinitesimally small relative to the market supply. Consequently, no individual can influence the asset price.

In this one-sided auction market, losses are possibly incurred when over-bidding occurs. From a behavioral finance perspective, one can interpret traders with low probability of overpredicting the fundamental value (implying a small number of upward biases) as relatively risk averse. For this reason, one may also interpret the θ-Assumption as an allowance for the entry of a sufficient number of buyers with a sufficient degree of risk aversion and with a positive probability (however small) of bidding arbitrarily close to the fundamental value. The presence of this group of buyers provides the driving force for the long run convergence.

Finally, the variation in the random shock about the fundamental value influences the extent to which efficiency occurs. The bigger is the variation in the random shock about the fundamental value, the bigger are the deviations of the asset price from the fundamental value.

Appendix A

This appendix consists of proofs for the results in Theorems 1–3 and Corollary 1. The proof in Theorem 1 makes use of the results in Lemmas 1, 3 and Proposition 1 in Appendix B.

The Proof of Theorem 1. Define buyer t's pseudo-price at time s, denoted by p_s^t, where $s \geqslant t,\ t \geqslant 1$, as the solution to the equation $\sum_{i=1,\, i \neq t}^{s} q_s^i(p_s^t) = 1$. That is, buyer t's pseudo-price at time s, p_s^t, would be the market clearing price at time s if buyer t were not in the market. In other words, it would be the market clearing price which would occur without the demand from buyer t.

Also define an indicator variable at time s for buyer t, denoted as l_s^t, where $s \geqslant t$, $t \geqslant 1$, to describe the position of buyer t's pseudo-price (p_s^t) relative to the fundamental value z_s at time s. Specifically, for any given $\varepsilon > 0$,

$$l_s^t = \begin{cases} 1 & \text{if } p_s^t < z_s - \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that if at any time s, the pseudo-price of any buyer is above $z_s - \varepsilon$ then the asset price is above $z_s - \varepsilon$. Use an indicator variable $\tilde{l}_s = 0$ to describe whether the pseudo-price of any buyer is above $z_s - \varepsilon$; otherwise $\tilde{l}_s = 1$. Specifically, for $s \geqslant t$, $t \geqslant 1$,

$$\tilde{l}_s = \prod_{t=1}^{s} l_s^t.$$

The following shows that

$$\Pr\left(\lim_{T \to \infty}\left(\frac{\sum_{s=1}^{T} \tilde{l}_s}{T}\right) = 0\right) = 1. \tag{A.1}$$

Define l_s^* to indicate whether the asset price is below $z_s - \varepsilon$ at time s, where $s \geqslant 1$. That is

$$l_s^* = \begin{cases} 1 & \text{if } p_s < z_s - \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

If $l_s^* = 1$ then the asset price at time s is below $z_s - \varepsilon$. Taking out any buyer's demand for the asset from the market demand function at time s means that the pseudo-price of that buyer is at least below $z_s - \varepsilon$. That is, $l_s^* = 1$ implies that $l_s^t = 1$ for all $t = 1, 2, \ldots, s$ or $\tilde{l}_s = 1$. This further implies that

$$\frac{\sum_{s=1}^{T} l_s^*}{T} \leqslant \frac{\sum_{s=1}^{T} \tilde{l}_s}{T}. \tag{A.2}$$

If Eq. (A.1) is true, then Eqs. (A.1) and (A.2) imply that

$$\Pr\left(\lim_{T \to \infty}\left(\frac{\sum_{s=1}^{T} l_s^*}{T}\right) = 0\right) = 1.$$

Therefore, Theorem 1 follows.

Now, the following proves that Eq. (A.1) is true. Before the proof of Eq. (A.1), some definitions of indicator variables are required. At time $s = 1, 2, \ldots, l_s$ is defined to indicate whether the asset price is below $z_s - \frac{2}{3}\varepsilon$.

$$l_s = \begin{cases} 1 & \text{if } p_s < z_s - \frac{2}{3}\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Define another random variable g_s^t to describe whether buyer t's prediction at time s is between $z_s - \frac{2}{3}\varepsilon$ and $z_s - \frac{1}{3}\varepsilon$. Specifically, define g_s^t, where $s \geqslant t$ and $t \geqslant 1$, as

$$g_s^t = \begin{cases} 1 & \text{if } -\frac{2}{3}\varepsilon \leqslant u_s^t \leqslant -\frac{1}{3}\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Now, Eq. (A.1) is shown by way of contradiction. Suppose not, then

$$\Pr\left(\text{there exist } a\ \varepsilon_0 > 0 \text{ and a subsequence } n_1, n_2, \ldots, n_T, \ldots,\right.$$
$$\left.\text{such that for all } T = 1, 2, \ldots, \frac{\sum_{s=1}^{n_T} \tilde{l}_s}{n_T} > \varepsilon_0\right) > 0. \tag{A.3}$$

Lemma 1 in Appendix B implies that for this ε_0 and ε; and for some positive $c < 1$,

$$\Pr\,(\text{there exists an infinite number of buyers in the set } \Theta(\varepsilon_0, \varepsilon, c)) = 1, \tag{A.4}$$

where

$$\Theta(\varepsilon_0, \varepsilon, c) = \left\{\text{buyer } t\text{: } \varepsilon_0 \theta_2^t \ln\left(\frac{\bar{z}}{(\bar{z} - \frac{2}{3}\varepsilon)}\right) + \theta_1^t \ln\left(\frac{\underline{z}}{(\bar{z} + u)}\right) > 0,\right.$$
$$\left.\theta_2^t \geqslant c,\ \theta_1^t + \theta_2^t + \theta_3^t = 1\right\}.$$

Now, consider buyer t in the set $\Theta(\varepsilon_0, \varepsilon, c)$. Since buyer t's pseudo-price at time i, p_i^t, is a function of $z_1, z_2, \ldots, z_i$; and $\omega_1, \omega_2, \ldots, \omega_{i-1}$; and $u_j^j, u_{j+1}^j, \ldots, u_i^j$, for $t \neq j \leqslant i$; and since buyer t's prediction error u_i^t is independent of $z_1, z_2, \ldots, z_i$; and $\omega_1, \omega_2, \ldots, \omega_{i-1}$; and $u_j^j, u_{j+1}^j, \ldots, u_i^j$, for $j \leqslant i$ and $j \neq t$; it follows that buyer t's prediction error is independent of his or her past and current pseudo-prices, that is, g_s^t is independent of l_i^t, for all $i \leqslant s$. Therefore, using Proposition 1 in Appendix B and using Eq. (A.4), it follows that

$$\lim_{T \to \infty} \Pr\left(\frac{\sum_{s=t}^{n_T} (l_s^t g_s^t)}{n_T - t + 1} > \frac{\mathrm{E}(g_s^t) \sum_{s=t}^{n_T} l_s^t}{n_T - t + 1} \middle| t \in \Theta(\varepsilon_0, \varepsilon, c)\right) = 1. \tag{A.5}$$

Using Bayes rule, Eqs. (A.4) and (A.5) imply that

$$\lim_{T \to \infty} \Pr\left(\frac{\sum_{s=t}^{n_T} (l_s^t g_s^t)}{n_T - t + 1} > \frac{\mathrm{E}(g_s^t) \sum_{s=t}^{n_T} l_s^t}{n_T - t + 1} \text{ for all } t \in \Theta(\varepsilon_0, \varepsilon, c)\right) = 1. \tag{A.6}$$

Since $\sum_{s=1}^{n_T} \tilde{l}_s / n_T > \varepsilon_0$, for all T, implies that, for sufficiently large T and for $t \in \Theta(\varepsilon_0, \varepsilon, c)$, $\sum_{s=t}^{n_T} l_s^t / (n_T - t + 1) > \varepsilon_0$ (due to $l_s^t \geqslant \tilde{l}_s$); and since $\mathrm{E}(g_s^t) = \theta_2^t$, applying Bayes rule to Eqs. (A.3) and (A.6), it follows that,

$$\lim_{T \to \infty} \Pr\left(\frac{\sum_{s=t}^{n_T} (l_s^t g_s^t)}{n_T - t + 1} > \varepsilon_0 \theta_2^t, \text{ for all } t \in \Theta(\varepsilon_0, \varepsilon, c) \middle| H(\varepsilon_0)\right) = 1, \tag{A.7}$$

where $H(\varepsilon_0)$ denotes the event that there exist a $\varepsilon_0 > 0$ and a subsequence $n_1, n_2, \ldots, n_T, \ldots$, such that $\sum_{s=1}^{n_T} \tilde{l}_s / n_T > \varepsilon_0$ for all $T = 1, 2, \ldots$.

Since buyer $t \in \Theta(\varepsilon_0, \varepsilon, c)$, it follows that $\theta_2^t \geqslant c$. This together with Eq. (A.7) also implies that there exists a further subsequence $n'_1, n'_2, \ldots, n'_T, \ldots$ of the sequence $n_1, n_2, \ldots, n_T, \ldots$, such that (see Durrett, 1991, p. 40, Theorem 6.2).

$$\Pr\left(\lim_{T \to \infty} \inf \left(\frac{\sum_{s=t}^{n'_T} (l_s^t g_s^t)}{n'_T - t + 1} \right) > \varepsilon_0 c, \text{ for all } t \in \Theta(\varepsilon_0, \varepsilon, c) \middle| H(\varepsilon_0) \right) = 1. \tag{A.8}$$

In addition, using Lemma 3 in Appendix B, it follows that for this $\varepsilon_0, \varepsilon$; and for some positive number $c < 1$,

$$\Pr\left(\lim_{T \to \infty} \left(\frac{\sum_{s=t}^{n_T} (l_s^t g_s^t l_s)}{n_T - t + 1} \right) = 0, \text{ for all } t \in \Theta(\varepsilon_0, \varepsilon, c) \right) = 1. \tag{A.9}$$

Again, applying Bayes rule to Eqs. (A.3) and (A.9), it follows that for this ε_0 and for some positive number $c < 1$,

$$\Pr\left(\lim_{T \to \infty} \left(\frac{\sum_{s=t}^{n_T} (l_s^t g_s^t l_s)}{n_T - t + 1} \right) = 0, \text{ for all } t \in \Theta(\varepsilon_0, \varepsilon, c) \middle| H(\varepsilon_0) \right) = 1. \tag{A.10}$$

The intuition behind Eq. (A.10) is that if $l_s^t g_s^t l_s = 1$, then one can show that buyer t makes a positive expected profit at time s. If Eq. (A.10) is not true, then one can construct a contradiction by showing that with positive probability that the wealth of buyer t who is in the set $\Theta(\varepsilon_0, \varepsilon, c)$ would grow exponentially while the wealth, coming from the initial endowment of buyers and coming from the market maker, is injected into the market arithmetically.

Applying Bayes rule to Eqs. (A.10) and (A.8), it follows that

$$\begin{aligned} \Pr\Bigg(&\left\{ \lim_{T \to \infty} \inf \frac{\sum_{s=t}^{n'_T} (l_s^t g_s^t)}{n'_T - t + 1} > \varepsilon_0 c, \text{ for all } t \in \Theta(\varepsilon_0, \varepsilon, c) \right\}, \\ &\left\{ \lim_{T \to \infty} \left(\frac{\sum_{s=t}^{n_T} (l_s^t g_s^t l_s)}{n_T - t + 1} \right) = 0, \text{ for all } t \in \Theta(\varepsilon_0, \varepsilon, c) \right\} \Bigg| H(\varepsilon_0) \Bigg) = 1. \end{aligned} \tag{A.11}$$

Since for buyer $t \in \Theta(\varepsilon_0, \varepsilon, c)$, $t \leqslant n'_T$,

$$\frac{\sum_{s=t}^{n'_T} (l_s^t g_s^t)}{n'_T - t + 1} = \frac{\sum_{s=t}^{n'_T} (l_s^t g_s^t l_s)}{n'_T - t + 1} + \frac{\sum_{s=t}^{n'_T} (l_s^t g_s^t (1 - l_s))}{n'_T - t + 1},$$

and using Eq. (A.11), it follows that

$$\Pr\left(\lim_{T \to \infty} \inf \left(\frac{\sum_{s=t}^{n'_T} (l_s^t g_s^t (1 - l_s))}{(n'_T - t + 1)} \right) > \varepsilon_0 c, \text{ for all } t \in \Theta(\varepsilon_0, \varepsilon, c) \middle| H(\varepsilon_0) \right) = 1. \tag{A.12}$$

Now, the remaining part of the proof consists of two steps. Step 1 shows that (using Eq. (A.12)) with positive probability, $(1-l_s)\sum_{t\in\Theta(\varepsilon_0,\varepsilon,c),\ t\leqslant s}(g_s^t l_s^t)$ is unbounded from above as $s\to\infty$. As a result, Step 2 shows that with a positive probability the expected total profits of the number of buyers equal to $\sum_{t\in\Theta(\varepsilon_0,\varepsilon,c),\ t\leqslant s}(g_s^t l_s^t)$ tends to infinity as $s\to\infty$. This contradicts the fact that the total profits earned by all buyers in any one time period must be bounded from above. Therefore, Eq. (A.1) must be true.

Step 1: One observation that can be drawn from Eq. (A.12) is that $(1-l_s)\sum_{t\in\Theta(\varepsilon_0,\varepsilon,c),\ t\leqslant s}(g_s^t l_s^t)$ must be unbounded as $s\to\infty$. That is,

$$\Pr\left(\text{for any given } M>0,\ \text{there exists a s}' \text{ such that, for some s} > s',\right.$$

$$\left.(1-l_s)\sum_{\substack{t\in\Theta(\varepsilon_0,\varepsilon,c)\\ t\leqslant s}}(g_s^t l_s^t)\geqslant M\,\middle|\,(H(\varepsilon_0),D_2)\right)=1, \tag{A.13}$$

where D_2 denotes the event that

$$\lim_{T\to\infty}\inf\left(\frac{\sum_{s=t}^{n'_T}(l_s^t g_s^t(1-l_s))}{(n'_T-t+1)}\right)>\varepsilon_0 c,\quad \text{for all } t\in\Theta(\varepsilon_0,\varepsilon,c).$$

Otherwise, there would be a contradiction. The contradiction can be seen from the following analysis. Suppose that Eq. (A.13) is not true, then

$$\Pr\left(\text{there exists a positive number, say } B>0,\ \text{such that,}\right.$$

$$\left.(1-l_s)\sum_{\substack{t\in\Theta(\varepsilon_0,\varepsilon,c)\\ t\leqslant s}}(g_s^t l_s^t)\leqslant B,\ \text{for all } s\,\middle|\,(H(\varepsilon_0),D_2)\right)>0. \tag{A.14}$$

Using Eqs. (A.12) and (A.14), it follows that

$$\Pr\left(\left\{\exists B>0,\ \text{s.t. for all } s,\ (1-l_s)\sum_{\substack{t\in\Theta(\varepsilon_0,\varepsilon,c)\\ t\leqslant s}}(g_s^t l_s^t)\leqslant B\right\}\ \text{and}\right.$$

$$\left\{\text{for all sufficiently large } T,\ \sum_{s=t}^{n'_T}(l_s^t(1-l_s)g_s^t)>\varepsilon_0 c(n'_T-t+1),\right.$$

$$\left.\left.\text{for all } t\in\Theta(\varepsilon_0,\varepsilon,c) \text{ and } t\leqslant n'_T\right\}\middle|H(\varepsilon_0)\right)>0. \tag{A.15}$$

Since $\Theta(\varepsilon_0, \varepsilon, c)$ contains an infinite number of buyers, select the first $\bar{N}$ buyers, $t_1, t_2, \ldots, t_{\bar{N}}$, where $\bar{N} > 2B/\varepsilon_0 c$, from the set $\Theta(\varepsilon_0, \varepsilon, c)$. If $(1 - l_s) \sum_{t \in \Theta(\varepsilon_0, \varepsilon, c),\ t \leqslant s} (g_s^t l_s^t) \leqslant B$, for all s then for $n'_T > t_{\bar{N}}$,

$$\sum_{s=1}^{n'_T} \left((1 - l_s) \sum_{k=1,2,\ldots,\bar{N},} (\tilde{l}_s^{t_k} \tilde{g}_s^{t_k}) \right) \leqslant B n'_T, \tag{A.16}$$

where $\tilde{l}_s^{t_k}$ and $\tilde{g}_s^{t_k}$ for $k = 1, 2, \ldots,$ and for $s = 1, 2, \ldots,$ are defined as

$$\tilde{l}_s^{t_k} = \begin{cases} l_s^{t_k} & \text{if } s \geqslant t_k \\ 0 & \text{if } s < t_k \end{cases} \quad \text{and} \quad \tilde{g}_s^{t_k} = \begin{cases} g_s^{t_k} & \text{if } s \geqslant t_k, \\ 0 & \text{if } s < t_k. \end{cases}$$

If $\sum_{s=t_k}^{n'_T} (l_s^{t_k}(1 - l_s) g_s^{t_k}) > \varepsilon_0 c (n'_T - t_k + 1)$, for all sufficiently large T and for all $t_k \leqslant n'_T$, then

$$\begin{aligned} \sum_{k=1}^{\bar{N}} \sum_{s=t_k}^{n'_T} (l_s^{t_k}(1 - l_s) g_s^{t_k}) &> \varepsilon_0 c \sum_{k=1}^{\bar{N}} (n'_T - t_k + 1) \\ &> \varepsilon_0 c \bar{N} (n'_T - t_{\bar{N}} + 1). \end{aligned} \tag{A.17}$$

Since

$$\sum_{s=1}^{n'_T} \left((1 - l_s) \sum_{k=1,2,\ldots,\bar{N},} (\tilde{g}_s^{t_k} \tilde{l}_s^{t_k}) \right) = \sum_{k=1}^{\bar{N}} \sum_{s=t_k}^{n'_T} (l_s^{t_k}(1 - l_s) g_s^{t_k})$$

and using Eqs. (A.4), (A.15), (A.16) and (A.17), it follows that

$$\Pr \left(\bar{N} < \lim_{T \to \infty} \left(\frac{B n'_T}{\varepsilon_0 c (n'_T - t_{\bar{N}} + 1)} \right) = \frac{B}{\varepsilon_0 c} \middle| H(\varepsilon_0) \right) > 0.$$

This further implies that

$$\Pr \left(\bar{N} < \frac{B}{\varepsilon_0 c} \right) > 0.$$

This contradicts the fact that $\bar{N} > 2B/\varepsilon_0 c$. Therefore, Eq. (A.13) must be true.

Now, Eq. (A.13) implies that

$$\Pr \Bigg(\text{for any given } M > 0, \text{ there exists } a\ s' \text{ such that, for some } s > s', \\ (1 - l_s) \sum_{\substack{t \in \Theta(\varepsilon_0, \varepsilon, c) \\ t \leqslant s}} (g_s^t l_s^t) \geqslant M \Bigg) > 0. \tag{A.18}$$

The intuition for Step 1 is as follows. Consider the infinite number of buyers in the set $\Theta(\varepsilon_0, \varepsilon, c)$. For each one of the buyers in the set, in order for the proportion of time that $l_s^t(1-l_s)g_s^t = 1$ to be bounded from below, then the total number of times, that the joint event of $l_s^t = 1$ for every buyer t and $g_s^t = 1$ for every buyer t whenever $l_s = 0$, must go to infinity as $s \to \infty$, which in turn implies that $(1-l_s) \sum_{t \in \Theta(\varepsilon_0, \varepsilon, c),\ t \leqslant s} (g_s^t l_s^t)$ is unbounded from above as $s \to \infty$.

Step 2: However, the following shows that if $(1-l_s) \sum_{t \in \Theta(\varepsilon_0, \varepsilon, c)\, t \leqslant s} (g_s^t l_s^t)$ is unbounded from above as $s \to \infty$, then there exists a time period at which the expected total profits of a finite number of buyers in the set $\Theta(\varepsilon_0, \varepsilon, c)$ exceeds the total profits that could be possibly made by all the buyers in any one time period. This is illustrated in the following analysis. Since both the liquidation value and the asset price are bounded and since the supply of the asset each time period is 1 unit, the maximum profits that all the buyers could possibly make in any one time period is bounded from above by $\bar{z}\exp(\bar{\omega})$.

Eq. (A.18) implies that for $M = 3(\bar{z}\exp(\bar{w}))/\varepsilon \underline{q}$,

$$\Pr\left(\exists s' \text{ s.t. for some } s > s', (1-l_s) \sum_{\substack{t \in \Theta(\varepsilon_0, \varepsilon, c) \\ t \leqslant s}} (g_s^t l_s^t) \geqslant \frac{3(\bar{z}\exp(\bar{w}))}{\varepsilon \underline{q}}\right) > 0. \tag{A.19}$$

Since $\Theta(\varepsilon_0, \varepsilon, c)$ contains an infinite number of buyers, select $\bar{M}$ buyers, $t_1', t_2', \ldots, t_{\bar{M}}'$, where $\bar{M} > 2B/\varepsilon_0 c$, from the set $\Theta(\varepsilon_0, \varepsilon, c)$. Hence, Eqs. (A.4) and (A.19) imply that

$$\Pr\Bigg(\exists s' \text{such that, for some } s > s',\ \exists\{t_1', t_2', \ldots, t_{\bar{M}}'\},\ \text{where } \bar{M} > \frac{3(\bar{z}\exp(\bar{w}))}{\varepsilon \underline{q}},$$
$$\text{s.t. } (g_s^{t_1'} l_s^{t_1'} = 1,\ g_s^{t_2'} l_s^{t_2'} = 1, \ldots, g_s^{t_{\bar{M}}'} l_s^{t_{\bar{M}}'} = 1) \text{ when } 1 - l_s = 1\Bigg) > 0. \tag{A.20}$$

Since at time period s, if $g_s^{t_k'} = 1$, $l_s^{t_k'} = 1$ and $(1-l_s) = 1$, then buyer t_k' bids in the interval $[z_s - \frac{2}{3}\varepsilon, z_s - \frac{1}{3}\varepsilon]$ while the pseudo-price of buyer t_k' is below $z_s - \varepsilon$ and the asset price is in the region $[z_s - \frac{2}{3}\varepsilon, z_s - \frac{1}{3}\varepsilon]$. This implies that buyer t_k' must be active and purchases at least a minimum quantity of shares of the asset ($\underline{q}$) which occurs in the extreme case where buyer t_k''s pseudo-price at time s, $p_s^{t_k'} = z_s - \varepsilon$ and the asset price at time s is $p_s = z_s - \frac{2}{3}\varepsilon$. (It can be shown that $\underline{q} = 1 - (\bar{z} - \varepsilon)/(\bar{z} - \frac{2}{3}\varepsilon)$.) Therefore, if for $k = 1, 2, \ldots, \bar{M}$, $g_s^{t_k'} = 1$, $l_s^{t_k'} = 1$ and $(1-l_s) = 1$, then

$$\sum_{k=1}^{\bar{M}} q_s^{t_k'} \geqslant \bar{M}\underline{q} \tag{A.21}$$

and

$$z_s - p_s \geqslant z_s - \left(z_s - \frac{1}{3}\varepsilon\right) = \frac{1}{3}\varepsilon. \tag{A.22}$$

Now consider the total expected profits of $\bar{M}$ buyers with $(g_s^{t_1'} l_s^{t_1'} = 1,\ g_s^{t_2'} l_s^{t_2'} = 1, \ldots, g_s^{t_{\bar{M}}'} l_s^{t_{\bar{M}}'} = 1)$ when $1 - l_s = 1$. Denote D_3 as a set of buyers $\{t_1', t_2', \ldots, t_{\bar{M}}'\}$, where $\bar{M} > 3(\bar{z}\exp(\bar{w}))/\varepsilon \underline{q}$, such that $(g_s^{t_1'} l_s^{t_1'} = 1,\ g_s^{t_2'} l_s^{t_2'} = 1, \ldots, g_s^{t_{\bar{M}}'} l_s^{t_{\bar{M}}'} = 1)$ when $1 - l_s = 1$. Using Eq. (A.21),

$$\begin{aligned} \mathrm{E}\left((v_s - p_s) \sum_{k \in D_3}^{\bar{M}} q_s^{t_k'} \right) &\geqslant \bar{M}\underline{q}\mathrm{E}(v_s - p_s) \\ &\geqslant \bar{M}\underline{q}\mathrm{E}(z_s - p_s) \\ &\geqslant \frac{1}{3}\varepsilon \bar{M}\underline{q}. \end{aligned} \tag{A.23}$$

The second to last inequality is due to $\mathrm{E}v_s \geqslant \mathrm{E}z_s$, (using $v_s = z_s \exp(\omega_s)$ and using Jensen's inequality). The last inequality is due to Eq. (A.22).

Using the definition of $\bar{M}$ and Eq. (A.23), it follows that

$$\mathrm{E}\left((v_s - p_s) \sum_{k \in D_3}^{\bar{M}} q_s^{t_k'} \right) > \bar{z}\exp(\bar{\omega}). \tag{A.24}$$

However, the maximum amount of total profits that could be earned by all the buyers in the market in any one time period is bounded from above by $\bar{z}\exp(\bar{\omega})$. This contradicts Eq. (A.24). Therefore Eq. (A.21) must be true. □

The Proof of Theorem 2. Consider the aggregate wealth of all the buyers at the end of time period T, which equals the aggregate gains that all the buyers have made up to time period T minus the aggregate losses that all the buyers have made up to time period T plus the aggregate initial wealth of all the buyers in the asset market at time T. That is,

$$\begin{aligned} \sum_{t=1}^{T} V_T^t &= \sum_{k=1}^{T} \left((v_k - p_k) \left(\sum_{t=1}^{k} q_k^t \right) \right) + \sum_{k=1}^{T} V_0 \\ &= \sum_{k=1}^{T} (v_k - p_k) + \sum_{k=1}^{T} V_0 \quad \left(\text{since } \sum_{t=1}^{k} q_k^t = 1 \right) \\ &= \sum_{k=1}^{T} (v_k - z_k) + \sum_{k=1}^{T} (z_k - p_k) + \sum_{k=1}^{T} V_0. \end{aligned} \tag{A.25}$$

To describe the position of p_t relative to z_t, define three indicator random variables $\hat{m}_t$, $\hat{l}_t$ and $\hat{a}_t$, for $t = 1, 2, \ldots$, as follows: for any given $\varepsilon > 0$,

$$\hat{m}_t = \begin{cases} 1 & \text{if } z_t \leqslant p_t \leqslant z_t + \varepsilon \\ 0 & \text{otherwise,} \end{cases}$$

$$\hat{l}_t = \begin{cases} 1 & \text{if } p_t < z_t \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \hat{a}_t = \begin{cases} 1 & \text{if } p_t > z_t + \varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\hat{l}_t + \hat{a}_t + \hat{m}_t = 1$ for any t. Eq. (A.25) can be rewritten as follows:

$$\frac{\sum_{t=1}^{T} V_T^t}{T} = \frac{\sum_{k=1}^{T} (v_k - z_k)}{T} + \frac{\sum_{k=1}^{T} (\hat{l}_k(z_k - p_k))}{T} + \frac{\sum_{k=1}^{T} (\hat{a}_k(z_k - p_k))}{T} + \frac{\sum_{k=1}^{T} (\hat{m}_k(z_k - p_k))}{T} + V_0. \tag{A.26}$$

Since when $\hat{m}_k = 1,\ z_k \leqslant p_k \leqslant z_k + \varepsilon$, it follows that

$$\sum_{k=1}^{T} (\hat{m}_k(z_k - p_k)) \leqslant 0. \tag{A.27}$$

Since when $\hat{l}_k = 1,\ p_k < z_k$ and furthermore, $z_k - p_k \leqslant \bar{z}$. Therefore,

$$\sum_{k=1}^{T} (\hat{l}_k(z_k - p_k)) \leqslant \bar{z} \sum_{k=1}^{T} \hat{l}_k. \tag{A.28}$$

Since when $\hat{a}_k = 1,\ p_k > z_k + \varepsilon$ and furthermore, $z_k - p_k < -\varepsilon$. Therefore,

$$\sum_{k=1}^{T} (\hat{a}_k(z_k - p_k)) \leqslant -\varepsilon \sum_{k=1}^{T} \hat{a}_k. \tag{A.29}$$

Using Eqs. (A.27)–(A.29), Eq. (A.26) becomes

$$\frac{\sum_{t=1}^{T} V_T^t}{T} \leqslant \bar{z}\frac{\sum_{k=1}^{T} \hat{l}_k}{T} - \varepsilon\frac{\sum_{k=1}^{T} \hat{a}_k}{T} + \frac{\sum_{k=1}^{T} (v_k - z_k)}{T} + V_0. \tag{A.30}$$

Since the asset price is strictly above zero and since all the buyers have initial endowments of wealth, it follows that for any T, $\sum_{t=1}^{T} V_T^t > 0$. This together with Eq. (A.30) implies that

$$\frac{\sum_{k=1}^{T} \hat{a}_k}{T} \leqslant \frac{\bar{z}}{\varepsilon}\frac{\sum_{k=1}^{T} \hat{l}_k}{T} + \frac{\sum_{k=1}^{T} (v_k - z_k)}{\varepsilon T} + \frac{V_0}{\varepsilon}. \tag{A.31}$$

Since

$$\frac{\sum_{k=1}^{T} (v_k - z_k)}{T} = \frac{\sum_{k=1}^{T}(z_k(\exp(\omega_k) - 1))}{T},$$

and denoting

$$I_{\omega_k>0} = \begin{cases} 1 & \text{if } \omega_k > 0, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that

$$\frac{\sum_{k=1}^{T} (v_k - z_k)}{T} \leqslant \frac{\sum_{k=1}^{T}(\bar{z}(\exp(\omega_k) - 1)I_{\omega_k>0})}{\#\{k\colon\ \omega_k > 0\}}\frac{\#\{k\colon\ \omega_k > 0\}}{T}. \tag{A.32}$$

Since $\{\omega_k\}_{k\geqslant 1}$ is an i.i.d random sequence, using the Strong Law of Large Numbers, it follows that with probability 1, as $T\to\infty$,

$$\frac{\sum_{k=1}^{T}(\bar{z}(\exp(\omega_k)-1)I_{\omega_k>0})}{\#\{k\colon\ \omega_k>0\}}\frac{\#\{k\colon\ \omega_k>0\}}{T} \to \bar{z}\,\mathrm{E}((\exp(\omega_k)-1)|\omega_k>0)\ \Pr\ (\omega_k>0). \tag{A.33}$$

Since $0<\Pr\ (\omega_k>0)<1$, Eqs. (A.32) and (A.33) imply that with probability 1 the following occurs: for any given $\varepsilon>0$, there exists a positive $\bar{k}$ such that for any i.i.d random shock sequence $\{\omega_k\}_{k\geqslant 1}$ with $\mathrm{E}((\exp(\omega_k)-1)|\omega_k>0)\leqslant\bar{k}$,

$$\lim_{T\to\infty}\left(\frac{\sum_{k=1}^{T}\ (v_k-z_k)}{T}\right)<\bar{z}\bar{k}\leqslant\frac{\varepsilon^2}{3}. \tag{A.34}$$

The result in Theorem 1 implies that

$$\Pr\ \left(\lim_{T\to\infty}\left(\frac{\sum_{k=1}^{T}\ \hat{l}_k}{T}\right)<\frac{\varepsilon^2}{3\bar{z}}\right)=1. \tag{A.35}$$

Set $\bar{V}_0\leqslant\varepsilon^2/3$. This together with Eqs. (A.31), (A.34) and (A.35) implies that with probability 1 the following occurs: for any given $\varepsilon>0$, there exist a positive $\bar{k}$ and a positive $\bar{V}_0$ such that, if $\mathrm{E}((\exp(\omega_t)-1)|\omega_t>0)\leqslant\bar{k}$ and if $V_0\leqslant\bar{V}_0$, there exists a time period T_2, such that for all $T>T_2$, $\sum_{k=1}^{T}\ \hat{a}_k/T<\varepsilon$. Therefore, the result in Theorem 2 follows. □

The Proof of Theorem 3. Notice that for any $T\geqslant 1$,

$$\frac{\#\{t\leqslant T\colon\ p_t<z_t-\varepsilon\}}{T}+\frac{\#\{t\leqslant T\colon\ p_t>z_t+\varepsilon\}}{T}+\frac{\#\{t\leqslant T\colon\ p_t\in[z_t-\varepsilon,z_t+\varepsilon]\}}{T}=1.$$

Therefore, using the results in Theorems 1 and 2 and letting $T^*=\max\{T_1,T_2\}$, Theorem 3 follows from the above equation. □

The Proof of Corollary 1. Consider buyer t's average wealth across time at the end of time period T.

$$\frac{V_T^t}{T}=\frac{V_0+\sum_{k=t}^{T}((v_k-p_k)q_k^t)}{T}. \tag{A.36}$$

Define two indicator random variables $\tilde{l}_t$ and $\tilde{a}_t$, for $t=1,2,\ldots$, as follows:

$$\tilde{l}_t=\begin{cases}1 & \text{if } p_t<z_t\\ 0 & \text{otherwise}\end{cases}\quad\text{and}\quad\tilde{a}_t=\begin{cases}1 & \text{if } p_t\geqslant z_t,\\ 0 & \text{otherwise.}\end{cases}$$

Notice that $\tilde{l}_t+\tilde{a}_t=1$ for any t. Using the above definitions, Eq. (A.36) becomes

$$\frac{V_T^t}{T}=\frac{\sum_{k=t}^{T}\ (\tilde{l}_k(v_k-p_k)q_k^t)}{T}+\frac{\sum_{k=t}^{T}\ (\tilde{a}_k(v_k-p_k)q_k^t)}{T}+\frac{V_0}{T}. \tag{A.37}$$

Since, when $\tilde{l}_k = 1$ at time k, $v_k - p_k = z_k \exp(\omega_k) - p_k < \bar{z}\exp(\bar{\omega})$, this together with $0 \leqslant q_k^t \leqslant 1$ implies that

$$\sum_{k=t}^{T} (\tilde{l}_k(v_k - p_k)q_k^t) < \bar{z}\exp(\bar{\omega}) \sum_{k=t}^{T} \tilde{l}_k. \tag{A.38}$$

Since, when $\tilde{a}_k = 1$, $v_k - p_k = z_k \exp(\omega_k) - p_k \leqslant z_k \exp(\omega_k) - z_k \leqslant \bar{z}(\exp(\bar{\omega}) - 1)$, this together with $0 \leqslant q_k^t \leqslant 1$ implies that

$$\sum_{k=t}^{T} (\tilde{a}_k(v_k - p_k)q_k^t) \leqslant \bar{z}(\exp(\bar{\omega}) - 1) \sum_{k=t}^{T} \tilde{a}_k. \tag{A.39}$$

Using Eqs. (A.38) and (A.39), Eq. (A.37) becomes

$$\frac{V_T^t}{T} < \bar{z}\exp(\bar{\omega}) \frac{\sum_{k=t}^{T} \tilde{l}_k}{T} + \bar{z}(\exp(\bar{\omega}) - 1) \frac{\sum_{k=t}^{T} \tilde{a}_k}{T} + \frac{V_0}{T}. \tag{A.40}$$

The result in Theorem 1 implies that, for any given $\varepsilon > 0$,

$$\Pr\left(\lim_{T\to\infty}\left(\frac{\bar{z}\exp(\bar{\omega})\sum_{k=t}^{T} \tilde{l}_k}{T}\right) \leqslant \frac{\varepsilon}{3} V_0\right) = 1. \tag{A.41}$$

The result in Theorem 2 implies that

$$\Pr\Bigg(\forall \varepsilon > 0, \exists \bar{k} > 0 \text{ and } \bar{V}_0 > 0 \text{ such that, if } \mathrm{E}((\exp(\omega_t) - 1)|\omega_t > 0) \leqslant \bar{k}.$$

$$\text{and if } V_0 \leqslant \bar{V}_0, \exists T_2', \text{ s.t. for } T > T_2',\ \bar{z}(\exp(\bar{\omega}) - 1)\frac{\sum_{k=t}^{T} \tilde{a}_k}{T} < \frac{\varepsilon}{3} V_0\Bigg) = 1. \tag{A.42}$$

Notice that $\lim_{T\to\infty} V_0/T < \frac{\varepsilon}{3} V_0$. This together with Eqs. (A.40), (A.41) and (A.42) implies that

$$\Pr\big(\forall \varepsilon > 0, \exists \bar{k} > 0 \text{ and } \exists \bar{V}_0 > 0 \text{ such that, if } \mathrm{E}((\exp(\omega_t) - 1)|\omega_t > 0) \leqslant \bar{k}$$

$$\text{and if } V_0 \leqslant \bar{V}_0,\ \exists T_3, \text{ s.t. for } T > T_3, V_T^t/T < \varepsilon V_0\big) = 1. \tag{A.43}$$

Now consider all the buyers' average wealth across times up to the end of time period T. Using Eq. (A.36) and since $\sum_{t=1}^{k} q_k^t = 1$, it follows that

$$\frac{\sum_{t=1}^{T} V_T^t}{T} = \frac{\sum_{k=1}^{T} V_0 + \sum_{k=1}^{T} (v_k - p_k)}{T}. \tag{A.44}$$

Since $\tilde{l}_k + \tilde{a}_k = 1$, Eq. (A.44) becomes

$$\frac{\sum_{t=1}^{T} V_T^t}{T} = \frac{\sum_{k=1}^{T} (\tilde{l}_k(v_k - p_k))}{T} + \frac{\sum_{k=1}^{T} (\tilde{a}_k(v_k - p_k))}{T} + V_0. \tag{A.45}$$

Since $\tilde{l}_k = 1$ implies that $z_k > p_k$ and $v_k - p_k > (\exp(\omega_k) - 1)z_k \geqslant (\exp(-\underline{\omega}) - 1)\bar{z}$, and since $\tilde{a}_k = 1$ implies that $v_k - p_k \geqslant \bar{z}\exp(-\underline{\omega}) - (\bar{z} + u)$, using Eq. (A.45), it follows that

$$\frac{\sum_{t=1}^{T} V_T^t}{T} \geqslant ((\exp(-\underline{\omega}) - 1)\bar{z})\frac{\sum_{k=1}^{T} \tilde{l}_k}{T} + (\bar{z}\exp(-\underline{\omega}) - (\bar{z} + u))\frac{\sum_{k=1}^{T} \tilde{a}_k}{T} + V_0. \tag{A.46}$$

Given that $(\exp(-\underline{\omega}) - 1)\bar{z} < 0$, Theorem 1 implies that with probability 1 the following occurs: for any $\varepsilon > 0$ and for any V_0,

$$\lim_{T\to\infty}\left(((\exp(-\underline{\omega}) - 1)\bar{z})\frac{\sum_{k=1}^{T} \tilde{l}_k}{T}\right) \geqslant -\frac{\varepsilon}{2}V_0. \tag{A.47}$$

Given that $\bar{z}\exp(-\underline{\omega}) - (\bar{z} + u) < 0$, the result in Theorem 2 implies that with probability 1 the following occurs: for any $\varepsilon > 0$, there exist a $\bar{k} > 0$ and a $\bar{V}_0 > 0$ such that, if $\mathrm{E}((\exp(\omega_k) - 1)|\omega_t > 0) \leqslant \bar{k}$ and if $V_0 \leqslant \bar{V}_0$,

$$\lim_{T\to\infty}\left((\bar{z}\exp(-\underline{\omega}) - (\bar{z} + u))\frac{\sum_{k=1}^{T} \tilde{a}_k}{T}\right) \geqslant -\frac{\varepsilon}{2}V_0. \tag{A.48}$$

Therefore Eqs. (A.46)–(A.48) imply that with probability 1 the following occurs: for any $\varepsilon > 0$, there exist a $\bar{k} > 0$ and a $\bar{V}_0 > 0$ such that, if $\mathrm{E}((\exp(\omega_k) - 1)|\omega_t > 0) \leqslant \bar{k}$ and if $V_0 \leqslant \bar{V}_0$,

$$\lim_{T\to\infty}\left(\frac{\sum_{t=1}^{T} V_T^t}{T}\right) \geqslant -\varepsilon V_0 + V_0. \tag{A.49}$$

Using Eqs. (A.43) and (A.49), it follows that with probability 1 the following occurs: for any $\varepsilon > 0$, there exist a $\bar{k} > 0$ and a $\bar{V}_0 > 0$ such that, if $\mathrm{E}((\exp(\omega_k) - 1)|\omega_t > 0) \leqslant \bar{k}$ and if $V_0 \leqslant \bar{V}_0$,

$$\lim_{T\to\infty}\left(\frac{V_T^t}{\sum_{t=1}^{T} V_T^t}\right) < \frac{\varepsilon V_0}{(1-\varepsilon)V_0}. \tag{A.50}$$

Let $\varepsilon' = \varepsilon/(1-\varepsilon)$, Corollary 1 follows from Eq. (A.50). $\square$

Appendix B

This appendix consists of three lemmas and one proposition. Lemmas 1 and 2 are used to establish Lemma 3. Proposition 1, Lemmas 1 and 3 are used to prove Theorem 1. Proposition 1 is also used to prove Lemma 2.

Proposition 1. Let $\{X_t\}_{t\geqslant 1}$ *be a random sequence with a finite expectation* $\mathrm{E}(X_t)$ *and* $|X_t|\leqslant M_1$, *where* M_1 *is a finite positive number. Let* Y_t, *where* $t = 1, 2, \ldots$, *be a random variable with* $|Y_t|\leqslant M_2$, *where* M_2 *is a finite positive number. If* X_t *is independent of* $Y_1, Y_2, \ldots, Y_t$, *then for any given* $\varepsilon > 0$,

$$\lim_{T\to\infty} \Pr\left(\left|\frac{\sum_{t=1}^{T}\ (X_t Y_t)}{T} - \frac{\sum_{t=1}^{T}(Y_t \mathrm{E}(X_t))}{T}\right| < \varepsilon\right) = 1.$$

Proof. Let $X_t = \mathrm{E}(X_t) + \varepsilon_t$. Since $\{X_t\}_{t\geqslant 1}$ is a random sequence with $\mathrm{E}(X_t) < \infty$ and $|X_t|\leqslant M_1$, it follows that $\{\varepsilon_t\}_{t\geqslant 1}$ is a random sequence with $\mathrm{E}\varepsilon_t = 0$ and $|\varepsilon_t|\leqslant M_1 < \infty$. Therefore, for all $t = 1, 2, \ldots$,

$$\frac{\sum_{t=1}^{T}(X_t Y_t)}{T} = \frac{\sum_{t=1}^{T}(Y_t \mathrm{E}(X_t))}{T} + \frac{\sum_{t=1}^{T}(\varepsilon_t Y_t)}{T}. \tag{B.1}$$

Consider the last term in the above equation $\sum_{t=1}^{T}(\varepsilon_t Y_t)/T$. Since X_t is independent of Y_t, ε_t is independent of Y_t, and it follows that

$$\mathrm{E}\left(\frac{\sum_{t=1}^{T}(\varepsilon_t Y_t)}{T}\right) = \frac{\sum_{t=1}^{T}(\mathrm{E}(\varepsilon_t)\mathrm{E}(Y_t))}{T} = 0. \tag{B.2}$$

Notice that

$$\begin{aligned} Var\left(\frac{\sum_{t=1}^{T}(\varepsilon_t Y_t)}{T}\right) = & \frac{\sum_{t=1}^{T}(Var(\varepsilon_t Y_t))}{T^2} + \frac{2}{T^2}\sum_{t=1}^{T}\ \sum_{s>t}^{T}(\mathrm{E}(\varepsilon_t Y_t \varepsilon_s Y_s) \\ & -\ \mathrm{E}(\varepsilon_t Y_t)\mathrm{E}(\varepsilon_s Y_s)). \end{aligned} \tag{B.3}$$

Since X_s is independent of $Y_1, \ldots, Y_s, \varepsilon_s$ is also independent of $Y_1, \ldots, Y_s$. Given $\mathrm{E}\varepsilon_t = 0$ for any t, it follows that the last term in the above equation is equal to zero. That is,

$$\frac{2}{T^2}\sum_{t=1}^{T}\ \sum_{s>t}^{T}\ (\mathrm{E}(\varepsilon_t Y_t \varepsilon_s Y_s) - \mathrm{E}(\varepsilon_t Y_t)\mathrm{E}(\varepsilon_s Y_s)) = 0. \tag{B.4}$$

Since $|\varepsilon_t|\leqslant M_1 < \infty$ and since $|Y_t|\leqslant M_2 < \infty$, it follows that $Var(\varepsilon_t Y_t)\leqslant M_3$, where M_3 is a finite positive number. Therefore,

$$0\leqslant\frac{1}{T^2}\sum_{t=1}^{T}\ (Var(\varepsilon_t Y_t))\leqslant\frac{1}{T}M_3. \tag{B.5}$$

Eqs. (B.3)–(B.5) imply that

$$Var\left(\frac{\sum_{t=1}^{T}(\varepsilon_t Y_t)}{T}\right)\to 0 \quad \text{as}\ T\to\infty. \tag{B.6}$$

Given Eqs. (B.2) and (B.6), using Markov's Theorem (see Gnedenko, 1962, p.232), it follows that for any given $\varepsilon > 0$,

$$\lim_{T\to\infty} \Pr\left(\left|\frac{\sum_{t=1}^{T}(\varepsilon_t Y_t)}{T}\right| < \varepsilon\right) = 1. \tag{B.7}$$

Using Eqs. (B.1) and (B.7), Proposition 1 follows. □

Lemma 1. For any given $\delta, \varepsilon > 0$ *and for some positive* $c < 1$,

Pr (There exists an infinite number of buyers in the set $\Theta(\delta, \varepsilon, c)) = 1$,

where $\Theta(\delta, \varepsilon, c) = \{\text{buyer } t\text{: } \delta\theta_2^t \ln(\bar{z}/(\bar{z} - \frac{2}{3}\varepsilon)) + \theta_1^t \ln(\underline{z}/(\bar{z} + u)) > 0,\ \theta_2^t \geqslant c,\ \theta_1^t + \theta_2^t + \theta_3^t = 1\}$.

Proof. As noted in the text, $\{(\theta_1^t, \theta_2^t)\}_{t\geqslant 1}$ is independently and identically distributed. Hence, define a random variable R_t, where $t = 1, 2, \ldots,$ as the following: for any given arbitrarily small $\delta, \varepsilon > 0$ and some positive $c < 1$,

$$R_t = \begin{cases} 1 & \text{if buyer } t \in \Theta(\delta, \varepsilon, c), \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\Theta(\delta, \varepsilon, c) = \{\text{buyer } t\text{: } \delta\theta_2^t \ln(\bar{z}/(\bar{z} - \tfrac{2}{3}\varepsilon)) + \theta_1^t \ln(\underline{z}/(\bar{z} + u)) > 0,$$
$$\theta_2^t \geqslant c, \theta_1^t + \theta_2^t + \theta_3^t = 1\} \neq \emptyset.$$

It follows that $\{R_t = 1\}_{t\geqslant 1}$ is also an i.i.d. random sequence. If the distribution function $F(\cdot)$ satisfies the θ-Assumption, then $\Pr(R_t = 1) = \Pr$ (buyer $t \in \Theta(\delta, \varepsilon, c)) = \int_{\delta\theta_2+\theta_1 D>0,\ \theta_2\geqslant c} \mathrm{d}F(\theta_1, \theta_2) > 0$, for any t, where

$$D = \frac{\ln\left(\dfrac{\underline{z}}{\bar{z}+u}\right)}{\ln\left(\dfrac{\bar{z}}{\bar{z} - \dfrac{2}{3}\varepsilon}\right)}.$$

Therefore,

$$\sum_{t=1}^{\infty} \Pr\ (R_t = 1) = \infty. \tag{B.8}$$

Using Eq. (B.8) and the Second Borel Cantelli Lemma (see Billingsley, 1995, p. 83),

$$\Pr\left(\sum_{t=1}^{\infty} R_t = \infty\right) = 1. \tag{B.9}$$

Eq. (B.9) further implies that for any given arbitrarily small $\delta, \varepsilon > 0$ and some positive $c < 1$,

$$\Pr\left(\text{There exists an infinite number of buyers in the set } \Theta(\delta, \varepsilon, c)\right) = 1.$$

Lemma 1 follows. $\square$

Lemma 2. Consider buyer t's wealth at time period T, V_T^t, and show that for any given $\varepsilon > 0$,

$$\lim_{T \to \infty} \Pr\left(\frac{\ln V_T^t}{T - t + 1} \geqslant \frac{\sum_{k=t}^{T}(l_k g_k^t l_k^t)}{T - t + 1} \ln\left(\frac{\bar{z}}{\bar{z} - \frac{2}{3}\varepsilon}\right) + \frac{\sum_{k=t}^{T} x_k^t}{T - t + 1} \ln\left(\frac{\underline{z}}{\bar{z} + u}\right) + \frac{\ln V_0}{T - t + 1}\right) = 1.$$

Proof. Consider buyer t's wealth at time period k, where $k \geqslant t$, $t = 1, 2, \ldots$,

$$V_k^t = V_{k-1}^t + (v_k - p_k) q_k^t. \tag{B.10}$$

If buyer t's prediction at time k coincides with the asset price, then buyer t is referred to as a marginal trader. In this case, buyer t's wealth may not be fully used up. Hence $q_k^t \in (0, V_{k-1}^t / p_k]$. Denote $q_k^t = r_k (V_{k-1}^t / p_k)$, where $r_k \in (0, 1]$. Clearly, r_k is independent of the random shock ω_i for all $i \geqslant k$. For the sake of the following discussion, define an indicator random variable M_k^t for $k \geqslant t$ to describe whether buyer t at time k is a marginal trader. That is,

$$M_k^t = \begin{cases} 1 & \text{if } b_k^t = p_k, \\ 0 & \text{otherwise.} \end{cases}$$

Define another random variable H_k^t, where $k \geqslant t$, to describe whether buyer t is active at time k, that is,

$$H_k^t = \begin{cases} 1 & \text{if } q_k^t > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Define random variables x_k^t and y_k^t, where $k \geqslant t$, to describe the position of buyer t's prediction error relative to $-\frac{2}{3}\varepsilon$ or $-\frac{1}{3}\varepsilon$ at time k. That is, for any given $\varepsilon > 0$,

$$y_k^t = \begin{cases} 1 & \text{if } u_k^t < -\frac{2}{3}\varepsilon \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad x_k^t = \begin{cases} 1 & \text{if } u_k^t > -\frac{1}{3}\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, define two random variables m_k and a_k, where $k \geqslant 1$, as the following:

$$m_k = \begin{cases} 1 & \text{if } z_k - \frac{2}{3}\varepsilon \leqslant p_k \leqslant z_k \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad a_k = \begin{cases} 1 & \text{if } p_k > z_k, \\ 0 & \text{otherwise.} \end{cases}$$

In addition to the above notation, the notation (i.e., l_k, g_k^t and l_k^t) defined in the proof of Theorem 1 is also used here.

Consider the following mutually exclusive cases where buyer t is a nonmarginal trader at time k, i.e. $b_k^t > p_k$:

(1) $l_k = 1$ and $g_k^t = 1$ meaning that $p_k < z_k - \frac{2}{3}\varepsilon$ and $-\frac{2}{3}\varepsilon \leqslant u_k^t \leqslant -\frac{1}{3}\varepsilon$;

(2) $l_k = 1$ and $x_k^t = 1$ meaning that $p_k < z_k - \frac{2}{3}\varepsilon$ and $u_k^t > -\frac{1}{3}\varepsilon$;

(3) $l_k = 1$, $y_k^t = 1$, $(1 - M_k^t) = 1$ and $H_k^t = 1$ meaning that $p_k < z_k - \frac{2}{3}\varepsilon$; $u_k^t < -\frac{2}{3}\varepsilon$; and buyer t is active at time k;

(4) $a_k = 1$, $x_k^t = 1$, $(1 - M_k^t) = 1$ and $H_k^t = 1$ meaning that $p_k > z_k$; $u_k^t > -\frac{1}{3}\varepsilon$; and buyer t is active at time k;

(5) $m_k = 1$, $x_k^t = 1$, $(1 - M_k^t) = 1$ and $H_k^t = 1$ meaning that $z_k - \frac{2}{3}\varepsilon \leqslant p_k \leqslant z_k$; $u_k^t > -\frac{1}{3}\varepsilon$; and buyer t is active at time k;

(6) $m_k = 1$, $g_k^t = 1$, $(1 - M_k^t) = 1$ and $H_k^t = 1$ meaning that $z_k - \frac{2}{3}\varepsilon \leqslant p_k \leqslant z_k$; $-\frac{2}{3}\varepsilon \leqslant u_k^t \leqslant -\frac{1}{3}\varepsilon$; and buyer t is active at time k.

In conclusion, for $b_k^t > p_k$, buyer t uses all of his or her wealth at time period k, i.e. $q_k^t = V_{k-1}^t / p_k$. Using Eq. (B.10), it follows that, if $b_k^t > p_k$,

$$V_k^t = \left(\frac{v_k}{p_k}\right)^{g_k^t l_k + x_k^t l_k + H_k^t (1-M_k^t) y_k^t l_k + H_k^t (1-M_k^t) x_k^t a_k + H_k^t (1-M_k^t) x_k^t m_k + H_k^t (1-M_k^t) g_k^t m_k} V_{k-1}^t. \tag{B.11}$$

Consider the following mutually exclusive cases where buyer t is a marginal trader at time k, i.e. $b_k^t = p_k$:

(7) $l_k = 1$, $y_k^t = 1$, $M_k^t = 1$ meaning that $p_k < z_k - \frac{2}{3}\varepsilon$; $u_k^t < -\frac{2}{3}\varepsilon$; and buyer t is a marginal trader at time k;

(8) $a_k = 1$, $x_k^t = 1$, $M_k^t = 1$ meaning that $p_k > z_k$; $u_k^t > -\frac{1}{3}\varepsilon$; and buyer t is a marginal trader at time k;

(9) $m_k = 1$, $x_k^t = 1$, $M_k^t = 1$ meaning that $z_k - \frac{2}{3}\varepsilon \leqslant p_k \leqslant z_k$; $u_k^t > -\frac{1}{3}\varepsilon$; and buyer t is a marginal trader at time k;

(10) $m_k = 1$, $g_k^t = 1$, $M_k^t = 1$ meaning that $z_k - \frac{2}{3}\varepsilon \leqslant p_k \leqslant z_k$; $-\frac{2}{3}\varepsilon \leqslant u_k^t \leqslant -\dfrac{1}{3}\varepsilon$; and buyer t is a marginal trader at time k.

In conclusion, for $b_k^t = p_k$, buyer t may not use up all of his or her wealth, hence, $q_k^t = (r_k V_{k-1}^t)/p_k$ where $r_k \in (0, 1]$. Using Eq. (B.10), it follows that, if $b_k^t = p_k$,

$$V_k^t = \left(1 - r_k + r_k \left(\frac{v_k}{p_k}\right)\right)^{M_k^t y_k^t l_k + M_k^t x_k^t a_k + M_k^t x_k^t m_k + M_k^t g_k^t m_k} V_{k-1}^t. \tag{B.12}$$

Therefore, at the end of time period k, bringing together Eqs. (B.11) and (B.12) produces

$$\begin{aligned} V_k^t = {} & \left(\frac{v_k}{p_k}\right)^{g_k^t l_k + x_k^t l_k + H_k^t (1-M_k^t) y_k^t l_k + H_k^t (1-M_k^t) x_k^t a_k + H_k^t (1-M_k^t) x_k^t m_k + H_k^t (1-M_k^t) g_k^t m_k} \\ & \times \left(1 - r_k + r_k \left(\frac{v_k}{p_k}\right)\right)^{M_k^t y_k^t l_k + M_k^t x_k^t a_k + M_k^t x_k^t m_k + M_k^t g_k^t m_k} V_{k-1}^t. \end{aligned}$$

One can show inductively that

$$V_T^t = \prod_{k=t}^{T} \left\{ \left(\frac{v_k}{p_k}\right)^{g_k^t l_k + x_k^t l_k + H_k^t(1-M_k^t) y_k^t l_k + H_k^t(1-M_k^t) x_k^t a_k + H_k^t(1-M_k^t) x_k^t m_k + H_k^t(1-M_k^t) g_k^t m_k} \right.$$
$$\left. \times \left(1 - r_k + r_k\left(\frac{v_k}{p_k}\right)\right)^{M_k^t y_k^t l_k + M_k^t x_k^t a_k + M_k^t x_k^t m_k + M_k^t g_k^t m_k} \right\} V_{t-1}^t.$$

Let $V_{t-1}^t = V_0$, take the logarithm of both sides, and divide by $T - t + 1$

$$\frac{\ln V_T^t}{T-t+1} = \frac{1}{T-t+1}\left\{ \sum_{k=t}^{T} \left([g_k^t l_k + x_k^t l_k + H_k^t(1-M_k^t) y_k^t l_k + H_k^t(1-M_k^t) x_k^t a_k \right.\right.$$
$$\left. + H_k^t(1-M_k^t) x_k^t m_k + H_k^t(1-M_k^t) g_k^t m_k] \ln\left(\frac{v_k}{p_k}\right)\right)$$
$$+ \sum_{k=t}^{T} \left[(M_k^t y_k^t l_k + M_k^t x_k^t a_k + M_k^t x_k^t m_k + M_k^t g_k^t m_k) \right.$$
$$\left.\left. \times \ln\left(1 - r_k + r_k\left(\frac{v_k}{p_k}\right)\right)\right] + \ln V_0 \right\}.$$

Furthermore, since $\ln(\cdot)$ is concave,

$$\ln\left(1 - r_k + r_k\left(\frac{v_k}{p_k}\right)\right) \geqslant (1 - r_k)\ln(1) + r_k \ln\left(\frac{v_k}{p_k}\right) = r_k \ln\left(\frac{v_k}{p_k}\right),$$

it follows that

$$\frac{1}{T-t+1} \ln V_T^t \geqslant \frac{1}{T-t+1}(A_{1T} + A_{2T} + A_{3T} + \ln V_0),$$

where

$$\left\{ \begin{array}{l} A_{1T} = \sum_{k=t}^{T} \left(g_k^t l_k \ln\left(\frac{v_k}{p_k}\right)\right), \\ A_{2T} = \sum_{k=t}^{T} \left([x_k^t l_k + H_k^t(1-M_k^t) y_k^t l_k + H_k^t(1-M_k^t) x_k^t m_k \right. \\ \qquad \left. + H_k^t(1-M_k^t) g_k^t m_k] \ln\left(\frac{v_k}{p_k}\right)\right) \\ \qquad + \sum_{k=t}^{T} \left[(M_k^t y_k^t l_k + M_k^t x_k^t m_k + M_k^t g_k^t m_k) r_k \ln\left(\frac{v_k}{p_k}\right)\right], \\ A_{3T} = \sum_{k=t}^{T} \left[H_k^t(1-M_k^t) x_k^t a_k \ln\left(\frac{v_k}{p_k}\right)\right] + \sum_{k=t}^{T} \left[M_k^t x_k^t a_k r_k \ln\left(\frac{v_k}{p_k}\right)\right]. \end{array} \right. \tag{B.13}$$

Now, the convergence of A_{1T}, A_{2T} and A_{3T} are shown, respectively, in Steps 1–3.

Step 1: Since the asset price at time k, p_k, is a function of $z_1, z_2, \ldots, z_k$; and $\omega_1, \omega_2, \ldots, \omega_{k-1}$; and $u_t^t, u_{t+1}^t, \ldots, u_k^t$, for all $t \leqslant k$; and since ω_k is independent of $z_1, z_2, \ldots, z_k$; and $\omega_1, \omega_2, \ldots, \omega_{k-1}$; and $u_t^t, u_{t+1}^t, \ldots, u_k^t$, for all $t \leqslant k$; it follows that ω_k is independent of p_k. This further implies that ω_k is independent of g_i^t and l_i, for all

$t \leqslant i \leqslant k$. This together with $\ln v_k = \ln z_k + \omega_k$, and Proposition 1 in Appendix B implies that

$$\frac{A_{1T}}{T-t+1} = \frac{\sum_{k=t}^{T} (g_k^t l_k((\ln z_k - \ln p_k) + \omega_k))}{T-t+1}$$
$$\rightarrow \frac{\sum_{k=t}^{T} (g_k^t l_k(\ln z_k - \ln p_k))}{T-t+1} + \frac{\sum_{k=t}^{T} (g_k^t l_k E(\omega_k))}{T-t+1} \quad \text{in probability.} \tag{B.14}$$

Since $l_k = 1$ implies that $p_k < z_k - \frac{2}{3}\varepsilon$, $\ln z_k - \ln p_k \geqslant \ln(z_k/(z_k - \frac{2}{3}\varepsilon)) \geqslant \ln(\bar{z}/(\bar{z} - \frac{2}{3}\varepsilon)) > 0$. Therefore, this together with $\mathrm{E}(\omega_k) = 0$, implies that Eq. (B.14) becomes

$$\lim_{T\to\infty} \Pr\left(\frac{A_{1T}}{T-t+1} \geqslant \frac{\sum_{k=t}^{T} (g_k^t l_k \ln(\bar{z}/(\bar{z} - \frac{2}{3}\varepsilon)))}{T-t+1}\right) = 1. \tag{B.15}$$

Step 2: For the same reasons as outlined at the beginning of Step 1, the random shock at time k, ω_k, is independent of $M_i^t, y_i^t, x_i^t, g_i^t, l_i, m_i, r_i, H_i^t$ for all $t \leqslant i \leqslant k$. This together with $\ln v_k = \ln z_k + \omega_k$ and Proposition 1 in Appendix B implies that,

$$\begin{aligned}\frac{A_{2T}}{T-t+1} = & \frac{1}{T-t+1}\sum_{k=t}^{T} ([x_k^t l_k + H_k^t(1 - M_k^t) y_k^t l_k + H_k^t(1 - M_k^t) x_k^t m_k \\ & + H_k^t(1 - M_k^t) g_k^t m_k + M_k^t y_k^t l_k r_k + M_k^t x_k^t m_k r_k + M_k^t g_k^t m_k r_k] \\ & \times (\ln z_k - \ln p_k + \omega_k)) \\ \rightarrow & \frac{1}{T-t+1}\sum_{k=t}^{T} ([x_k^t l_k + H_k^t(1 - M_k^t) y_k^t l_k + H_k^t(1 - M_k^t) x_k^t m_k \\ & + H_k^t(1 - M_k^t) g_k^t m_k + M_k^t y_k^t l_k r_k + M_k^t x_k^t m_k r_k + M_k^t g_k^t m_k r_k] \\ & \times (\ln z_k - \ln p_k + E(\omega_k))) \quad \text{in probability.}\end{aligned} \tag{B.16}$$

Now, since $\mathrm{E}(\omega_k) = 0$, and since the equation $l_k = 1$ or $m_k = 1$ implies that $p_k \leqslant z_k$, it follows that if $l_k = 1$ or $m_k = 1$, then $\ln z_k - \ln p_k \geqslant 0$. Therefore, Eq. (B.16) becomes

$$\lim_{T\to\infty} \Pr\left(\frac{A_{2T}}{T-t+1} \geqslant 0\right) = 1. \tag{B.17}$$

Step 3: Similarly, since $\ln v_k = \ln z_k + \omega_k$ and since ω_k is independent of H_i^t, M_i^t, x_i^t, a_i, and r_i for all $t \leqslant i \leqslant k$, using Proposition 1 in Appendix B,

$$\frac{A_{3T}}{T-t+1} \rightarrow \frac{\sum_{k=t}^{T} \{(H_k^t(1 - M_k^t) x_k^t a_k + M_k^t x_k^t a_k r_k)((\ln z_k - \ln p_k) + E(\omega_k))\}}{T-t+1} \quad \text{in probability.} \tag{B.18}$$

Since $a_k = 1$ implies $p_k > z_k$, it follows that if $a_k = 1$, then $0 > \ln(z_k/(p_k)) > \ln(\underline{z}/(\bar{z} + u))$; and furthermore, $H_k^t(1 - M_k^t) x_k^t a_k + M_k^t x_k^t a_k r_k \leqslant x_k^t$ and $\mathrm{E}(\omega_k) = 0$. Therefore, this together with Eq. (B.18) implies that

$$\lim_{T\to\infty} \Pr\left(\frac{A_{3T}}{T-t+1} \geqslant \frac{\sum_{k=t}^{T}\{x_k^t \ln(\underline{z}/(\bar{z}+u))\}}{T-t+1}\right) = 1. \tag{B.19}$$

Using Eqs. (B.13), (B.15), (B.17) and (B.19), it follows that

$$\lim_{T\to\infty} \Pr\left(\frac{\ln V_T^t}{T-t+1} \geqslant \frac{\sum_{k=t}^{T}\{g_k^t l_k \ln(\bar{z}/(\bar{z}-\frac{2}{3}\varepsilon)) + x_k^t \ln(\underline{z}/(\bar{z}+u))\} + \ln V_0}{T-t+1}\right) = 1. \tag{B.20}$$

Since $g_k^t l_k \geqslant g_k^t l_k l_k^t$, using Eq. (B.20), Lemma 2 follows. $\square$

Lemma 3. *For any given* $\delta, \varepsilon > 0$, *for some positive number* $c < 1$ *and for any subsequence* $n_1, n_2, \ldots, n_T, \ldots,$

$$\Pr\left(\lim_{T\to\infty}\left(\frac{\sum_{s=t}^{n_T}(l_s^t g_s^t l_s)}{n_T-t+1}\right) = 0, \text{ for all } t \in \Theta(\delta,\varepsilon,c)\right) = 1,$$

where $\Theta(\delta,\varepsilon,c)$ *is defined in Lemma* 1. *That is*,

$$\Theta(\delta,\varepsilon,c) = \left\{\text{buyer } t\text{: } \delta\theta_2^t \ln\left(\frac{\bar{z}}{\bar{z}-\frac{2}{3}\varepsilon}\right) + \theta_1^t \ln\left(\frac{\underline{z}}{\bar{z}+u}\right) > 0, \quad \theta_2^t \geqslant c,\ \theta_1^t+\theta_2^t+\theta_3^t = 1\right\}.$$

Proof. Lemma 1 implies that for any given $\delta, \varepsilon > 0$ and for some positive $c < 1$,

$$\Pr(\text{there exists an infinite number of buyers in the set } \Theta(\delta,\varepsilon,c)) = 1, \tag{B.21}$$

where

$$\Theta(\delta,\varepsilon,c) = \left\{\text{buyer } t\text{: } \delta\theta_2^t \ln\left(\frac{\bar{z}}{\bar{z}-\frac{2}{3}\varepsilon}\right) + \theta_1^t \ln\left(\frac{\underline{z}}{\bar{z}+u}\right) > 0, \quad \theta_2^t \geqslant c,\ \theta_1^t+\theta_2^t+\theta_3^t = 1\right\}.$$

Now, consider buyer t in the set $\Theta(\delta,\varepsilon,c)$. If the following equation is true,

$$\Pr\left(\lim_{T\to\infty}\left(\frac{\sum_{s=t}^{n_T}(l_s^t g_s^t l_s)}{n_T-t+1}\right) = 0 \,\middle|\, t \in \Theta(\delta,\varepsilon,c)\right) = 1, \tag{B.22}$$

then using Bayes rule, Eqs. (B.21) and (B.22) implies that

$$\Pr\left(\lim_{T\to\infty}\left(\frac{\sum_{s=t}^{n_T}(l_s^t g_s^t l_s)}{n_T-t+1}\right) = 0, \text{ for all } t \in \Theta(\delta,\varepsilon,c)\right) = 1.$$

Therefore, Lemma 3 follows. Now, the following proves Eq. (B.22). The proof is shown by way of contradiction. Suppose not, then

$$\Pr\Bigg(\text{there exist a } \delta_0>0 \text{ and a further subsequence } n_1^0, n_2^0, \ldots, n_T^0, \ldots. \\ \text{of the subsequence } n_1, n_2, \ldots, n_T, \ldots, \text{ such that} \\ \frac{\sum_{s=t}^{n_T^0}(l_s^t g_s^t l_s)}{n_T^0-t+1} > \delta_0\theta_2^t, \text{ for all } T \geqslant 1 \,\Bigg|\, t\in\Theta(\delta,\varepsilon,c)\Bigg) > \delta_0. \tag{B.23}$$

Lemma 2 implies that for buyer $t\in\Theta(\delta,\varepsilon,c)$ and at time period n_T^0,

$$\lim_{T\to\infty}\Pr\left(\frac{\ln V_{n_T^0}^t}{n_T^0-t+1} \geqslant \frac{\sum_{s=t}^{n_T^0}(l_s^t g_s^t l_s)}{n_T^0-t+1}\ln\left(\frac{\bar{z}}{\bar{z}-\frac{2}{3}\varepsilon}\right) + \frac{\sum_{s=t}^{n_T^0} x_s^t}{n_T^0-t+1}\ln\left(\frac{\underline{z}}{\bar{z}+u}\right) + \frac{\ln V_0}{n_T^0-t+1}\,\middle|\, t\in\Theta(\delta,\varepsilon,c)\right) = 1. \tag{B.24}$$

Consider buyer $t\in\Theta(\delta,\varepsilon,c)$. Since $\{u_s^t\}_{s\geqslant t}$ is an i.i.d. random sequence, $\{x_s^t\}_{s\geqslant t}$ is an i.i.d. random sequence. The Strong Law of Large numbers implies that given $t\in\Theta(\delta,\varepsilon,c)$,

$$\frac{\sum_{s=t}^{n_T^0} x_s^t}{n_T^0-t+1} \to \theta_1^t \quad \text{as } T\to\infty \text{ with probability 1.} \tag{B.25}$$

Eqs. (B.23)–(B.25) imply that

$$\lim_{T\to\infty}\Pr\left(\frac{\ln V_{n_T^0}^t}{n_T^0-t+1} \geqslant \delta\theta_2^t\ln\left(\frac{\bar{z}}{\bar{z}-\frac{2}{3}\varepsilon}\right) + \theta_1^t\ln\left(\frac{\underline{z}}{\bar{z}+u}\right)\,\middle|\, t\in\Theta(\delta,\varepsilon,c)\right) > 0. \tag{B.26}$$

Since $t\in\Theta(\delta,\varepsilon,c)$ implies that for some positive λ,

$$\delta\theta_2^t\ln\left(\frac{\bar{z}}{\bar{z}-\frac{2}{3}\varepsilon}\right) + \theta_1^t\ln\left(\frac{\underline{z}}{\bar{z}+u}\right) > \lambda,$$

this together with Eq. (B.26) implies that

$$\lim_{T\to\infty}\Pr\left(\frac{\ln V_{n_T^0}^t}{n_T^0-t+1} \geqslant \lambda \,\middle|\, t\in\Theta(\delta,\varepsilon,c)\right) > 0.$$

This together with Eq. (B.21) further implies that for any given $F>0$,

$$\lim_{T\to\infty}\Pr\left(\frac{V_{n_T^0}^t}{n_T^0} > F\right) > 0. \tag{B.27}$$

However, since the aggregate profits of all the buyers in any one time period is bounded from above, the average total wealth of all the buyers across time is also bounded from above. It follows that the average total wealth of any one buyer across time is bounded from above. This contradicts Eq. (B.27).

Therefore, Eq. (B.22) must be true. □

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